DISCUSSED CLINICAL CASES

eOftalmo

Refraction of elliptical surfaces

Refração em superfícies elípticas

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KEYWORDS:

Aspherical refraction; Elliptical surfaces; Multifocality.

ABSTRACT

Refraction from elliptical surface sections is analyzed in this study. With specific adjustments of their eccentricities, the condition of aspherical refraction (image formation at an only point the focal point of the surface can be achieved when the incident wave exhibits plane wavefronts and propagates parallel to the optical axis, i.e., when the object (radiant power source) is at infinite distance. On the contrary, for objects located at finite distances, elliptical surfaces cannot produce aspherical refraction. The formulations and their respective demonstrative calculations are presented in this study. We also show that multifocality, governed by varying radii of curvature on a surface, is a specific optical condition of elliptical sections which serve as matrices of these refractometric applications.

DESCRITORES:

Refração esférica; Superfícies elípticas; Multifocalidade.

RESUMO

Considera-se a refração em secções de superfícies elípticas para mostrar que com ajustamentos específicos de suas excentricidades se possa prover a condição de “asfericidade” (formação da imagem em um único ponto, o *foco imagem* da superfície) quando a incidência for a de frentes de ondas planas e com direção paralela ao eixo óptico, isto é, quando o objeto (fonte de energia radiante) estiver à distância infinita. Contrariamente, para objetos situados a distâncias finitas, a “asfericidade” não pode ser assegurada por superfícies elípticas. As formulações e os respectivos cálculos demonstrativos são apresentados. Também se mostra que a *multifocalidade,* dependente de raios de curvatura variáveis em uma superfície, é condição óptica própria das secções elípticas que servem como matrizes dessas aplicações refratométricas.

INTRODUCTION

The study of optical geometry is of paramount importance in ophthalmology, since it addresses one of the most commonly used aspects in its daily practice: eye refractometry with its possible resulting applications (prescription of optical corrections). In fact, it is recognized that eye consultations, for the most part, occur due to problems that can be corrected by the proper application of simple measures related to eye refraction, such as eyeglass, contact, and intraocular lenses; surgical remodeling of the anterior face of the cornea; and other interventions. Thus, in the two most important books published by the Brazilian Council of Ophthalmology on the subject, the emphasis given to the optical phenomenon of refraction and its applications, in the first sentence of their prefaces, is paradigmatic. The oldest one[1] starts by quoting that *“Eye refractometry is the most demanding procedure among all who take a person to eye consultation.”* The more recent book[2] reiterates this importance in

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the following: *"Among the multiple actions expected of an ophthalmologist in the exercise of his profession, the most common is optical prescriptions."*

Obviously, knowledge about refraction is not limited to such specialists. It extends to those who develop the principles for these solutions as well as to those who manufacture or sell the products that make them possible. This knowledge is also valuable for other diverse areas of human activity, such as visual prospection of far distant objects (astronomical or terrestrial telescopy) or those very small (microscopy). In fact, until recently, refraction was regarded as an optical phenomenon linked to the propagation of visible light. Currently, however, refraction is known to be more general, extending to radiations of any frequency.

Despite not being an exclusively optical phenomenon, refraction was first described in such connotation. In fact, the empirical observations obtained in the 2nd century by Ptolemy (Klaúdius Ptolemaios who lived from 90-168 AC according to some sources, or 100-160 AC according to others) are surprising. His measurements of refractive angles from air to water and air to glass, are remarkable approximations of the exact values, especially considering the relatively precarious instrumentation he used to obtain those results. For example, for every 10° increments from 0° to 60°, from air to water, his absolute errors are between 0 to 39′ (at the incidence of 20°), (16.14′±16.48′). From air to glass, the errors were between 0 to 34′ (at the incidence of 20°), (17.12±15.11) , with an overall average error margin of only 2.08%. Almost all differences are accounted by the fact that measured values exceed the real exact values, which we could attribute to a systematic error in the measurement process. It is therefore quite possible that if Ptolemy had the mathematical arsenal by which the law of refraction was later enunciated, he would have come to its formulation. He correctly postulated that when light (or rays of vision) penetrate a medium with a higher refractive index, it approaches the perpendicular to the surface; contrarily, when it travels from that medium to another medium possessing a lower refractive index (air), it distances itself from that perpendicular (Hirschberg, p. 151)[3], a principle that remains unchanged.

It took another millennium and a half since then for the generic law of refraction-a concept with elegant simplicity-to be formulated once the mathematical instrumental necessary to enunciate this knowledge was unveiled in the 17th century. It addresses the possible change of direction of propagation of radiant energy on a medium toward that of the refracted radiant energy in the next, at each specific point of a surface separating them.

This discovery is attributed to Willebrord Snel van Royen (1580-1626), or Willebrord Snellius (with two "l"s in the Latin transcription of his Dutch name, due to which the Anglo-Saxon version of his name "Snell" became widespread). Without a specific date to certify it (being discovered only after his death) Snell wrote the following:

$$\mathbf{c_i . \csc i = c_r . \csc r} \quad \text{(F. 01)}$$

where the cosecants of the incidence (csc i) and refraction (csc r) angles are inversely proportional to the speeds of light in their respective media (c_i or c_r). The canonical form, as it is now known, was proposed by René Descartes (1637) :

$$\mathbf{n_i . \sin i = n_r . \sin r} \quad \text{(F. 02)}$$

where the sine of the incidence (sin i) and refractive (sin r) angles are inversely proportional to the refractive indices of their respective mediums. The refractive index of a medium (such as that of incidence, n_i) corresponds to the speed of light value in that medium (such as that of incidence, c_i) relative to the speed of light in vacuum (c_0), i.e., $n_i = c_0 / c_i$. Therefore, a value of inverse magnitude, implying a lower speed of light in the medium, results in a higher refractive index. Given that the cosecant of an angle is the reciprocal trigonometric function of the respective sine and considering the refractive index is the reciprocal of the speed of light, there is an absolute reciprocity between the formulations of Snell and Descartes, which substantiates using the expression "Snell-Descartes" law:

$$\mathbf{(\csc r) / (\csc i) = (c_i / c_r) = [(c_0 / c_r) / (c_0 / c_i)] = n_r / n_i = (\sin i) / (\sin r)} \quad \text{(F.03)}$$

The phenomenon of refraction, therefore, is governed by a principle formulated using the relationship of angles (expressed by a trigonometric function), which is universally reproducible for the pair of mediums between which it occurs, i.e., the relationship between its refraction indices remains constant. Quantifying these angles of incidence, *i*, and of refraction, *r*, implicitly requires the establishment of a reference, which is the imaginary normal line or the perpendicular to the point where refraction occurs.

In geometry, perpendicular lines to a *point* is nonsensical (infinite straight lines in any directions can pass through a point). To solve this question, one considers that the point where the refraction occurs is the place where, both, the surface (which has the point) and a tangent to it coincide. So that, a perpendicular to the tangent surface at this point is also perpendicular to the surface at that point. In other words, among the infinite lines which could serve as reference, only one, the "normal line" to the surface is valid. Therefore, although refraction is, effectively, a punctual phenomenon, it depends on the *curvature of the surface* at the point where it occurs.In other words, the claim that refraction at a point is independent of the surface is only partially correct. In its essence, the assertion is true: a light ray's *direction* (direction of the wave front with which radiant energy is propagated) remains the same regardless of whether the surface is flat, concave, convex, or in any other shape (Figure 1). However, this statement does not apply to one of the main practical applications of optics, in determining the *image position* of an object (Figure 1).

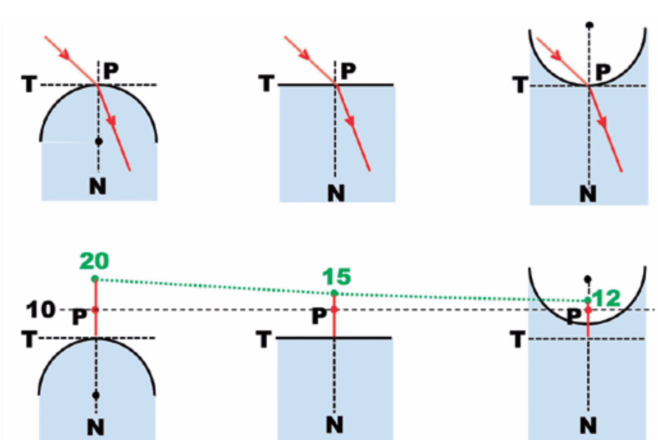

Figure 1. Representation of refraction in a plane where the propagation of radiation is considered at a point (P) of a surface and of a tangent plane to it (T). Therefore, the perpendicular line to the plane, at P, is also perpendicular to the surface, that is, its "normal line" (N). On a convex, flat, or concave surface, a medium of incidence with a lower refractive index (above, $n_i = 1.0$) and a higher index of the medium of refraction (below, $n_r = 1.5$) produce identical refraction angles (r) for the same incidence angles (i). Although the relationship between them (*i* and *r*) might be different, the ratio between their sines always follows refraction's law. It is therefore said that there is no difference between "refraction" at a point, regardless of a surface's shape: be it convex, flat, concave, or any other. However, for the image *position*, the conditions differ (below).

E.g.: for an object (P) located at distance p= -10 cm, if we assume the radii of curvature for the convex (R= +20 cm), flat (R= +∞), and concave (R = -20) surfaces, their respective images will form at distances q = -20, -15 and -12 cm, i.e., in *different positions* relative to each of the surfaces, dependent on the respective surface *curvatures* at the point considered. If the object was at -100 cm, the positions of the images are +100, -150 and -42.86 cm respectively.

Curvature of a surface

Although refraction is postulated as occurring at a "*point*", the correct claim is that it shall be measured relatively to a perpendicular line to it. Which refers to either a surface, a section of it (in three-dimensional space), or to another (tangent) line (in two-dimensional space). In other words, the treatment of a perpendicular to a *point on a surface* (or *line*) is subjected to the geometry of a surface's c*urvature* (or line) at that specific point, and this perpendicular is quantified by the *radius of curvature* of a surface (or line) at that specific point. Essentially, the concept of refraction is closely dependent on a surface's (or line's) *curvature* at one of its points.

Lines or surfaces may be mathematically defined by their curvatures, or by their reciprocal curvature radii: either constant and infinite (that of a straight line, or a plane), constant and finite (those of circles, or spheres), or progressively variables (e.g., those of conical sections, spheroids, or ellipsoids), among other natures. Refraction on flat surfaces has been widely studied in optical prisms. On the other hand, ophthalmic lenses for correcting ocular optical defects are all curved (one of their faces, at least). Nevertheless, it is odd that their refraction principles may be explained simply by the principles of spherical surfaces (given the relative simplicity of their geometry). Moreover, although it is recognized that the study of spherical optics is inherently defective, given the *spherical aberration* phenomenon,the approach of alternative curves is not commonly offered. But elliptical surface optics would provide the solutions to a*spherical* lens optics (that is, without presenting *spherical aberration*) and optical refinements as those of *multifocal* lenses.

In fact, a thorough study of curvatures imposes great difficulties such as requiring knowledge of differential geometry. In certain simpler cases, such as refraction on elliptical surfaces, the Cartesian analytical treatment of problems would suffice. Even so, the discomfort of following these paths is notorious, as wisely exposed in the presentation of another highly celebrated national textbook[4], alluding to an undeniable pedagogical reference on the study of refraction[5]: *"fortunately Donders was self-admittedly no mathematician and he wrote in clear and simple language, so that his book became popular."* And also, in subsequent complementation, the author of another classic work[6], "Duke-Elder's *Practice of Refraction" (1st Edition 1928)*, sought to avoid a mathematical presentation of refraction errors and the way to correct them[4].

Unequivocally, mathematical developments are generally difficult to follow up when they are unfamiliar, but indispensable to those who want to deepen any subject on which they are founded (the celebrated Helmholtz, in his celebrated three volumes of the Treaty on Physiological Optics, overused this resource). This does not justify the lack of reference to the nature of the solution of the problem. The lack of any mention of terms such as *"ellipse"* and *"elliptical curve (or surface)"* or similar terms is absolutely disconcerting, as is any correlation to the optics presented either in books on the study of refraction-those discussed earlier[1-6], others[7-10], those dedicated to optics (lenses manufacture)[11,12], or even in the most well-known international literature-old and recent alike [13-25]. When used, these terms have little in common with the refraction phenomenon's own nature; e.g., the *Tscherning ellipse*[24] (a graphical expression of the relationship between the dioptric power of ophthalmic lenses and the recommended base curves), or *elliptical polarization*[19] (in relation to one of the modalities of the electromagnetic energy polarization phenomenon). In fact, even when elliptical sections are considered, such as sections of cylindrical (or toric) lenses inclined relative to their axes, the optics of elliptical sections are not described . In fact, one of the most important ophthalmological compendiums[26] states that *"The surface dealt with in optics are generally spherical in shape … while the production of non-spherical or asphericsurfaces (which are actually more desirable in some instances) is extremely difficult. For this reason, spherical surfaces are employed almost exclusively in optics, and they usually perform adequately"* (pp. 6-7)[26]. Even when addressing cylindrical lenses, no reference is made to "ellipticals" (for their sections) On the contrary, little familiarity with such elementary geometric issues leads to the surprising statement that *"There is no optical power in oblique meridians of a toric refracting surface"* (p. 46, op. cit.). The mistake was not to consider the coplanarity of the incident and refracted rays, but to consider *"…'skew rays' because they are skewed about the axis,"* with which the text concludes as follows: *"these limiting rays are deviated by different amounts in the* ***two corresponding directions****"* (The bold type was my addition). In a later edition[27] this oversight is corrected, acknowledging that *"There is a different power in every meridian"* (p. 87), which is illustrated by an enlightening figure (fig. 72), despite it not detailing how and why this refraction (on elliptical surfaces) may occur.

In the revered and deep treatise on physiological optics by Helmholtz, there is a reference to *"ellipsoidal refracting surface"* (volume I, p. 194)[13], but its content is subject to aberrations occurring in refraction by the surface of an ellipsoid (such as that of an astigmatic cornea), describing the geometric figure that is modernly known as the *"Sturm's Conoid."* In fact, Helmholtz duly credits this author in the previous chapter of the same volume, although followed by a corrective remark[14]: *"Aberrations of the kind that Sturm supposes actually do seem to occur in most human eyes, and the phenomena dependent on them will be described; where, however, it will be shown that the interval between the two focal planes is by no means so important as STURM thinks and that, instead of promoting the clearness of vision, this defect in the eye tends rather to impair it."* It should be noted that even Allvar Gullstrand-who was awarded the Nobel Prize in Physiology or Medicine in 1911, precisely for his studies on ocular dioptrics-invited to review this monumental work by Helmholtz and to enrich it with the insertion of his perspicacious notes and appendices, does not dwell on the subject, nor does it mention ellipses and their optics[28].

In short, the lack of explicit references to refraction on elliptical surfaces appears to justify its' succinct presentation, in order to eventually arouse interests in its' further deepening.

Refraction by spherical surfaces

Spheres offer great simplicity in the study of refraction, given their curvature remains constant at any point, i.e., a single center through which all normal lines pass, which are also perpendicular to any tangents of its surface. The flat sections of a sphere are circles with different radii of curvature, the maximum value of which corresponds to that of the equatorial diameter.

The fact that there exists only one center (C) allows relative measurement of its radius of curvature (R) from its surface and the distances from the center to an object point (p) and that to its respective image, as a function of the angle of incidence (i). Note that because one considers a point object, regardless of its position in space relative to the spherical surface, the object can always be considered to be on the same line that contains its center of curvature, i.e., on a respective optical axis. Figure 2 shows that applying the law of sines to OPC and CIP triangles respectively results in

$$[OP/ \sin (i - a)] = [OC/ \sin (180 - i)]$$
$$[IP / \sin [180 - (i - a)] = [IC / \sin r]$$
Given $\sin [180 - (i - a)] = \sin (i - a)$ and
$\sin (180 - i) = \sin i$, $(OP \sin i) / OC = IP \sin r / IC \rightarrow$
$\sin (180 - i) = \sin i$, then
$$(OP \sin i) = (IP \sin r) / IC \rightarrow$$
$$(n_r / n_i) = (IP.\ OC)/(OP.\ IC) \quad (F.\ 04)$$

This geometric representation (F. 04) can be transformed into the variables of their respective meanings, i.e., the distances *p, s, and R* and the coordinates of point P, since

$$IP = [y^2 + (s - R + x)^2]^{½},\ OC = (p + R),$$
$$OP = [y^2 + (p + R - x)^2]^{½},\ IC = (s - R).$$

Thus, $(nr - ni) = \{[y^2 + (s - R)^2 + 2 (s - R).x + x^2]^{½}.\ (p + R)\} / \{[y^2 + (p + R)^2 - 2 (p + R).x + x^2]^{½}.\ (s - R)\}$

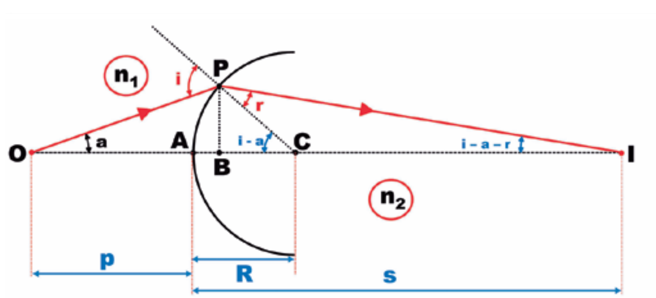

Figure 2. Schematic representation of refraction at a point P over a spherical surface's section with center in C and curvature radius PC = AC = R. The radiation has a source object (O) whose distance from point A of the surface is AO = p. The image of O forms at position I, whose distance to the surface's apex (A) is AI = s. The coordinates of P(x, y) are given by x = BC, y = PB.

From the equation of circles, $x^2 + y^2 = R^2$; thus,

$$\boldsymbol{(n_r/n_i) = \{[R^2 + (s - R)^2 + 2\ (s - R).(R^2 - y^2)^{½}]^{½}.\ (p + R)\} / \{[R^2 + (p + R)^2 - 2\ (p + R).(R^2 - y^2)^{½}\ ]^{½}\ (s - R)\}} \quad (F.\ 05)$$

i.e., given the values of the variables *y* (point of incidence "height," relative to the optical axis) and *R* (radius of a surface's curvature), a second-degree equation can be derived to calculate *s* (depending on p), or of *p* (as a function of *s*).

In any case, this formulation, although generic, is avoided. Instead, the paraxial ray equation is preferred, which implies a negligible incidence angle value (incidence "coincident" to the optical axis). In fact, if $y = 0$,

$$(n_r / n_i) = \{[R^2 + (s\text{-}R)^2 + 2\ (s - R).R]^{½}\ (p + R)\}/ \{[R^2 + (p + R)^2 - 2\ (p + R).R]^{½}(s - R)\}$$
$$\rightarrow \mathbf{(n_r / n_i) = [s\ (p + R)] / [p\ (s - R)]} \quad (F.\ 06)$$

Even more common is the formula resulting from a development of the above.

$$p\ (s - R).\ n_r = n_i.\ s\ (p + R) \rightarrow$$
$$(n_r - n_i)\ s.\ p = R\ [(s.\ n_i) + (p.\ n_r)] \rightarrow$$
$$[(n_r - n_i) / R] = [(s.\ n_i) / (s.\ p)] + [(p.\ n_r) / (s.\ p)] \rightarrow$$
$$\mathbf{[(n_r - n_i) / R] = F = (n_i / p) + (n_r / s)} \quad (F.\ 07)$$

An F-value is a surface's focal power and is expressed in the optical unit *diopters* (with the symbol D) when the radius of curvature (R) is quantified in meters. It is *constant* for each spherical surface for it depends on invariable values: the radius of curvature of the surface (R) and the refractive indices of the media separated by it (n_i, of the incidence medium, and n_r, of the refractive medium).

Another simplification of the generic formula (F. 05) is to consider an object's position at an infinite distance. Hence,

$$(n_r / n_i)^2 = \{[R^2 + (s - R)^2 + 2\ (s - R).(R^2 - y^2)^{½}]^{½} / (s - R)^2$$

a quadratic equation for the value $(s - R)$, dependent on the variable *y*. Solving the second-degree equation:

$$\mathbf{(s - R)\ [(n_r)^2 - (n_i)^2] / n_i = (n_i).\ (R^2 - y^2)^{½} + [(R.\ n_r)^2 - (y.\ n_i)^2]^{½}} \quad (F.\ 08)$$

In this case, $y = R \sin i$:

$$\mathbf{(s - R)\ [(n_r)^2 - (n_i)^2] / R.\ n_i = (n_i.\ \cos i) + [(n_r)^2 - (n_i.\ \sin i)^2]^{½}} \quad (F.\ 09)$$

This formula is very important because it shows that rays parallel to the optical axis (i.e., effectively coming from an infinite distance) reach the surface at different heights relative to the optical axis *(y)* and, or, with different values of the angle of incidence *(i)*, will produce images at different distances *(s)* from the surface, that is, a *longitudinal spherical aberration*. For example, table 1 shows that *s* values depend on those of *y*, for $n_r = 1.5$, $n_i = 1.0$, R = 20 cm.

In other words, energy reaching a surface does not concentrate on a single point (focus) but rather disperses over a space whose boundaries are contained by a surface called *caustic* (Figure 3).

Table 1. Distances from image position (s) relative to a spherical surface with radius of curvature (R) equal to 20 cm, separating the incidence medium (n_i = 1.0) and refractive medium (n_r = 1.5) according to the angle of incidence (i). Incident rays are considered parallel to the optical axis.

Angle of incidence (i)	Image position (s)
0º	60.000
10º	59.596
20º	58.403
30º	56.484
40º	53.941
50º	50.919
60º	47.596
70º	44.179
80º	40.881
90º	37.889

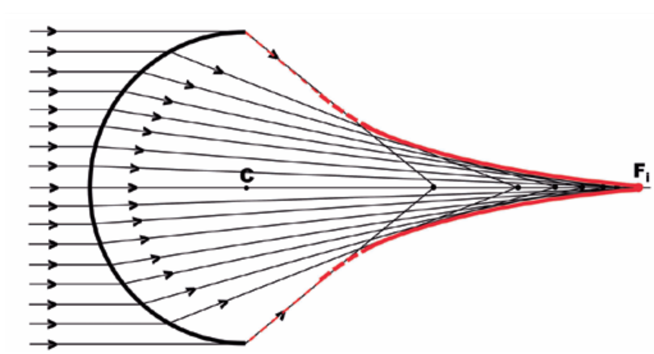

Figure 3. Representation of refraction on a spherical surface section. Incident rays parallel to the optical axis (object at an infinite distance) project images at different distances from the surface: the closer to it, the further away from the optical axis are the respective incident rays. Because of this "spherical aberration", the incident energy disperses in a space delimitated by a new surface (called *caustic)* pictorially shown as a red line, in the section considered. Caustic is the envelope of infinite images (formed by the crossing of refracted rays) and formed by the crossing of refracted rays originating from infinitely close incident rays.

Aspherical surfaces

The undesirable condition of *spherical aberration* makes it desirable to seek another surface that can concentrate refracted energy at a single point (the *image focal point,* that of the image distance of an object situated on the optical axis and infinite distance; s = 60 cm Table 1 example), all the energy incident on the surface. This is equivalent to searching for surfaces whose radii of curvature are progressively larger, i.e., less sharp curvatures. In other words, determining the specific radius of curvature *(R_y)* for each height relative to the optical axis *(y)* could enable the image of the object to coincide with the image corresponding to the incidence i = 0°.

Therefore, it boils down to rearranging the same F. 09, but by switching the sought variable, which is the value of the curvature radius *(Ry)* such that the image position is always the same (s = 60 cm, Table 2). To facilitate the calculation of R according to the other variables, F. 09 must be rearranged to

$$\mathbf{R_y = [n_r\ (n_r + n_i)\ R_0] \,/\, \{(n_r)^2 - (n_i)^2\ (1 - \cos i) + n_i\ [(n_r)^2 - (n_i.\ \sin i)^2]^{1/2}\}} \quad \text{(F.10)}$$

Figure 4 illustrates tables 1 and 2, which show the variation of an image's position according to the height relative to the optical axis in which the incidence of a radius is parallel to it (Figure 4A); further, it conversely shows the curvature radius variation required for an image to always form on the same point (Figure 4B).

Table 2. Relationship of the different radii of curvature of a surface separating media with refractive indices n_i = 1.0 and n_r = 1.5 for each incidence of rays parallel to the optical axis, such that the distance from the image to the surface is identical (s = 60 cm) to that produced by zero incidence (R_0 = 20 cm).

Angle of incidence (i)	Radius of curvature (R_y)
0º	20.000 cm
10º	20.136 cm
20º	20.547cm
30º	21.245 cm
40º	22.246 cm
50º	23.567 cm
60º	25.212 cm
70º	27.162 cm
80º	29.353 cm
90º	31.672 cm

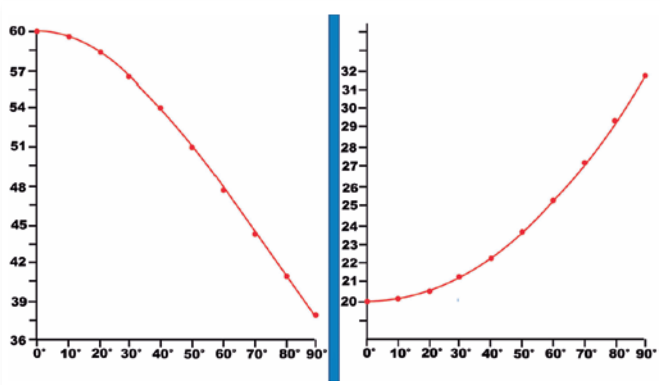

Figure 4. (A) Left: representation of the relationship between angles of incidence of rays parallel to the optical axis at different heights (abscissa) and the distance from the surface at which the respective image is formed (ordinate), on a spherical surface of radius of curvature 20 cm, separating media with refractive indices n_i = 1.0 and n_r = 1.5. (B) Right, representation of the relationship between incidence angles parallel to the optical axis at different heights of it (abscissa) and the radius of curvature of the surface at the respective points, so that the image is always formed over the same optical axis location (ordinate) (Data in Table 2).

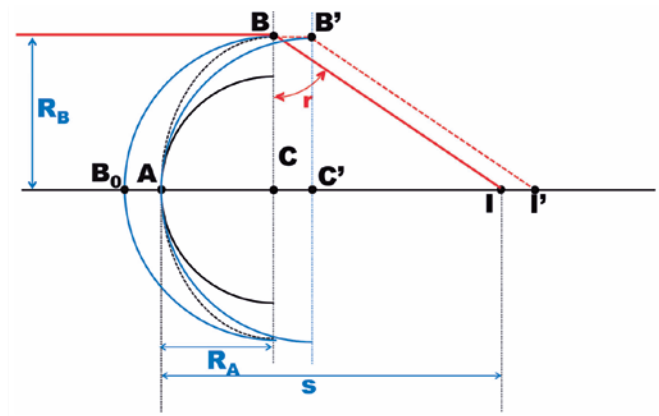

Figure 5. Representation of two curves, with the same center C, one with radius of curvature AC = R_A and the other with radius of curvature B_0C (R_B, greater than R_A), such that the refracted rays : (a) of an object at infinite distance, perpendicular to point A of the first and (b) tangent to point B of the second, produce an image in the same optical axis location (I). If the curves were coincidental to their apexes, the curvature center of the second would move to position C' and the image of point B' would be formed at I'

The underlying principle of table 2 raises some remarks, as explained in figure 5. In fact, the distance at which the image of a null incidence (i = 0°) formed by the surface with center C and radius of curvature AC = R_A is formed is AI = s. A surface with a greater radius of curvature, R_B = BC, with the same center of surface A, forms the image for incidence i = 90° at the same location (I). However, for this to occur, its apex (the position in which the surface is cut by the optical axis) must be at B_0, a place which does not coincide with that of the apex of the anterior surface that generated the first image (A). If there these apices coincided (in A), the curve of that surface would describe arc AB′, its center of curvature would not be C, but C′ and the position of the image would be at I′ (Figure 5). Thus, in order for the apices to coincide (at A), as much as the centers of curvature (at C), the drawing of curves with images formed in the same location (I), without the so-called "spherical aberration," must follow **another** curve (line interrupted between A and B in Figure 5).

In fact, there are several curves with increasing or decreasing progressions of the curvature radii at each of its points. Ellipses are those that enable the adjustment of refractive variables (refractive indices between the media separated by the surface and variable radii of curvature) such that refracted rays are always formed at the same "focal" point (Figure 6).

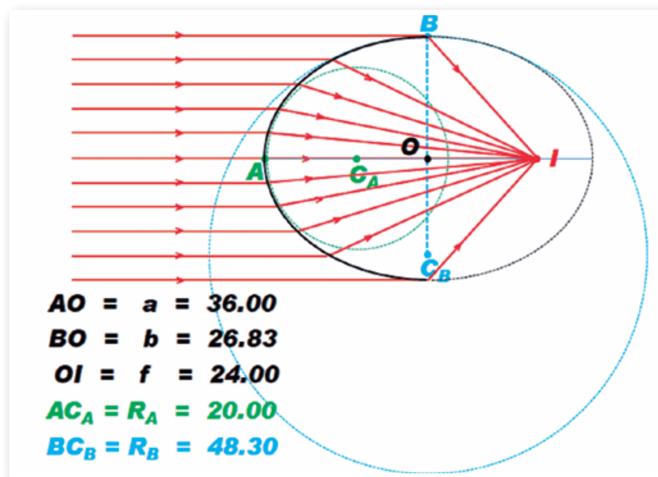

Figure 6. Schematic depiction of the refraction of incident rays parallel to the optical axis (AOI line) of an elliptical surface (black continuous stroke in the left half and dotted on the right), separating media with refractive index n_i = 1.0 and n_r = 1.5. Vertex A of the ellipse is the one with the highest curvature (shortest radius), corresponding to an imaginary circle with a center in C_A; vertex B is the one with least curvature (greater radius) corresponding to an imaginary circle with center in C_B. in this case, the points of the ellipse surface have progressive intermediate curvature radii, between them, (or "regressively," in another condition). All refracted rays cross point I (focus image of the elliptical surface).

In fact, unlike spherical surfaces in which refraction aberration appears, precisely because of their constant radius of curvature, on elliptical surfaces an increasing or decreasing progression of these radii of curvature can be observed (Figure 7), determining either an accentuation of those aberrations, or their reduction until they are canceled and reverted. Thus,

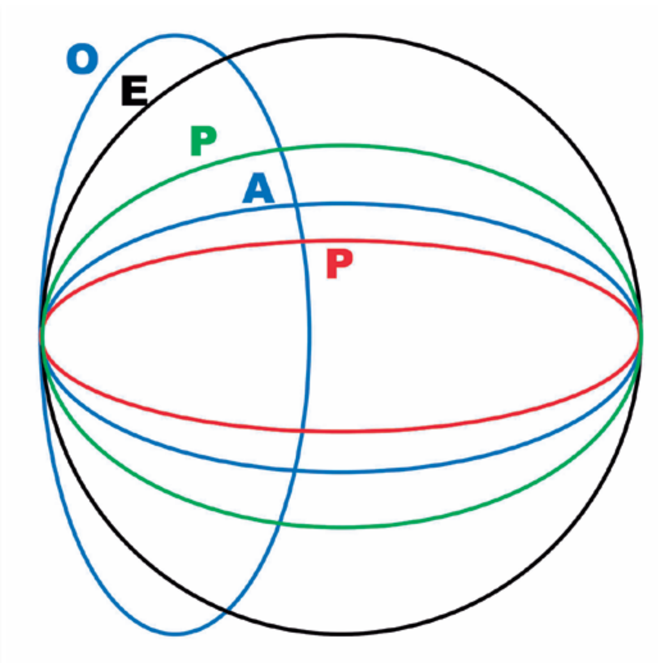

Figure 7. Family of closed curves, ellipses (O, A and the two named P) and circle (E). One of them corresponds to an aspherical curve (A, longer horizontal axis) which, is identical to "O" (longer vertical axis), only its presentation's position varies . Curve "O" is called an *oblate* whereas "A" and "P" are *prolate* forms. (The curves between "A" and "E," such as the green "P," are simply called *prolates*; "A" is an *aspherical prolate*; whereas those with even more flattened shapes, such as the red "P," are called *hyperprolates*.)

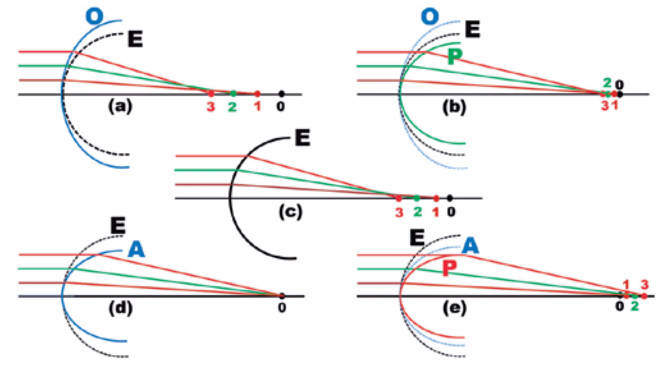

Figure 8. Schematic representation of refraction by different surfaces, separating an incidence medium (with a lower refractive index) from another (with a higher index). (c): In spherical (E), aberration is considered *positive*: the greater the distance between the optical axis and the incident rays parallel to it, that is, the more peripheral the incident rays are, the closer to the surface their respective images are. (a): On a surface which curvature radii are progressively *shorter* for incidence points occurring farther from the optical axis (O), this aberration (positive) increases. (b): The aberration decreases when the curvature radii become progressively *longer*, (d): it is annulled in a specific condition (no spherical aberration); and (e): it is reversed, in relation to the aberration of the spherical surface, i.e., it presents a *negative aberration*. The most "open" curves (O) are called *oblates* whereas the most "closed" are called *prolates*, among which is the *aspherical* (A).

a *single specific* shape (for each pair of optical medium separated by the surface) is the one that relates to the condition of *non-aberration of sphericity*, although strictly speaking, all of them are aspherical, that is, *non spherical* (Figure 8).

One of the distinctive mathematical traits of the ellipses is their *eccentricity*-a value that corresponds to their deformation relative to the circle. (The circle's eccentricity is zero.) However, the eccentricity of an ellipse cannot be directly associated with an aberration type, since the same ellipse (thus, the same eccentricity) can relate to the optical axis in several ways. In fact, ellipses O and A (Figures 7 and 8) are identical in their eccentricities, but differ when relating to the optical axis, with the longest of their axes in the horizontal plane (curve A) or to the smallest (curve O), which effects on refraction are completely different. Note that the inverse occurs for the curvature radius of the horizontal apical point: the shortest curvature radius is that of curve A and the longest is that of O.

The semantic issue of asphericity

Obviously, whether by increasing or decreasing variations of the curvature radii, all resulting curves are "non-spherical", or "aspherical".All curves produce refractional aberration by the surface's shape, with the exception of one. This special optical surface is termed as "non-aberrant" or "inaberrant." Thus, the most appropriate terminologies to define the type of aberration are *superaberration* (positive) when this aberration is increased with respect to *spherical aberration* (curve O, Figure 8a), *subaberration* when it is reduced (P, Figure 8b), simply "inaberration" when there is no aberration (curve A, Figure 8d), and *contraberration* (negative) when it reverses direction (curve P, Figure 8e). Oblate curves are always *superaberrant*, whereas *prolates* may be *subaberrant* (Figure 8b), *inaberrant* (Figure 8d), or *contraberrant* (Figure 8e). There is therefore no reason to say that *inaberrant* surfaces are prolate, for despite this being true, it represents a particular case: not every prolate curve is inaberrant.

The geometry of ellipses

Other elements of the ellipse geometry are not shown in figure 9, but in figure 10. In ellipses, becau-

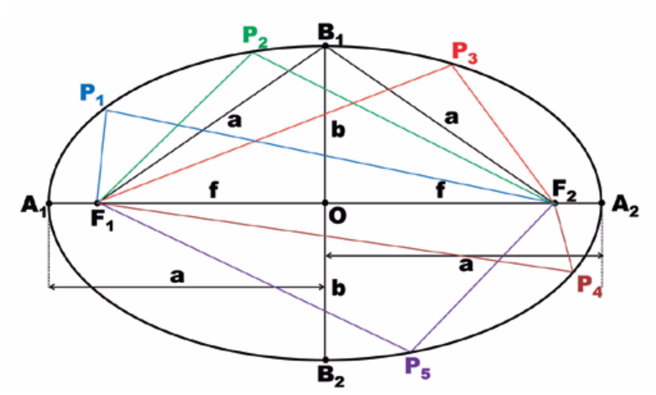

Figure 9. Pictorial representation of an ellipse, with its main elements: A_1 and A_2, vertices of the major axis (A_1A_2 = 2 a); A_1O = OA_2 = a, semimajor axes; B_1 and B_2, the minor axis' vertices (B_1B_2 = 2 b); B_1O = OB_2 = b, semiminor axes ; F_1 andF_2, foci of the ellipse; F_1F_2 = 2 f, interfocal length; F_1O = OF_2 = f, interfocal semidistances. The distances between the foci and the respective apices of the minor axes coincide with the length of the longer semiaxis (F_1B_1 = B_1F_2 = F_2B_2 = B_2F_1 = a). The sum of the distances from any point of the ellipse (e.g., P_1, P_2, P_3, P_4, P_5, or any other) to both foci (F_1 and F_2) always equals the major axis' length (F_1P_1 + P_1F_2 = F_1P_2 + P_2F_2 = F_1P_3 + P_3F_2 = F_1P_4 + P_4F_2 = F_1P_5 + P_5F_2 = 2 a). Point "O" at the middle of the major and minor axes is the center of the ellipse.

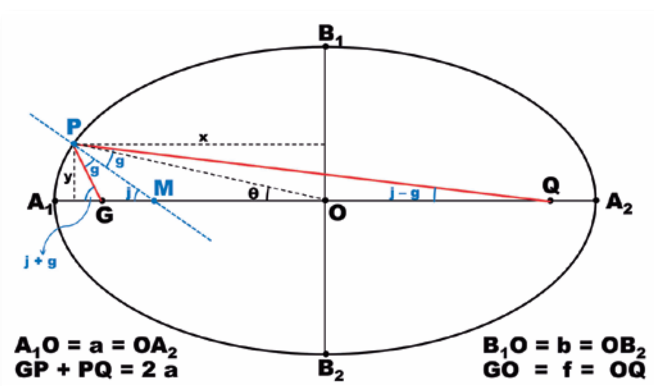

Figure 10. Representation of the ellipse's geometric relationships. The normal at one point on the surface (P) is the bisector of the angle formed between the lines connecting it to each of the foci (G and Q).

se the distance between B_1 (or B_2) and one of the foci (G or Q) equals the length of the major semiaxis *(a)*, the following relationship between variables *a, b,* and *f* becomes evident:

$$a^2 = b^2 + f^2 \quad \text{(F. 11)}$$

The ellipse's canonical equation, which relates the Cartesian coordinates of one of its points P(x, y) in relation to the origin of the measurement system (simplifiedly taken as the center of the ellipse), can be deduced from figure 9's elements as follows:

$$(x / a)^2 + (y / b)^2 = 1 \quad \text{(F.12)}$$

Another relationship is the one defining the ellipse's eccentricity (e):

$$e = f / a \quad \text{(F. 13)}$$

This value (of eccentricity, *e*) distinguishes the so-called *conical sections*, namely

(a) The circle (e = 0) obtained by the flat section of the cone perpendicular to its main axis.

b) The ellipses (1 > e > 0) obtained by the flat section of the cone inclined relative to its main axis but crossing it.

c) The parabola (e = 1) obtained by the flat section of the cone, parallel to one of its geratrices (lines connecting the apex to the base).

d) The hyperbolas (e > 1) obtained by the flat section of the cone, parallel, or inclined to its main axis, but without crossing it.

The relationship of the Cartesian coordinates of a point P (x, y) in a system which origin is the center of the O ellipse (0.0) is given by

$$OP = [x^2 + y^2]^{½} = (y / \sin θ) = (x / \cos θ) \rightarrow$$
$$y.\ \cos θ = x.\ \sin θ \quad \text{(F. 14)}$$

By replacing *y* of F. 14 in the canonical ellipse equation (F. 12), we get

$$x^2.\ b^2 + y^2.\ a^2 = a^2.\ b^2 \rightarrow$$
$$x^2.\ b^2 + a^2[(x^2.\ \sin^2 θ) / (\cos^2 θ)] = a^2.\ b^2 \rightarrow$$
$$x^2\ [(b^2 \cos^2 θ) + (a^2 \sin^2 θ)] = (a^2.\ b^2) \cos^2 θ \rightarrow$$
$$x = (a.\ b.\ \cos θ) / [(b^2 \cos^2 θ) + (a^2 \sin^2 θ)]^{½} \quad \text{(F. 15)}$$

Similarly,

$$y = (a.\ b.\ \sin θ) / [(b^2 \cos^2 θ) + (a^2 \sin^2 θ)]^{½} \quad \text{(F. 16)}$$

Therefore, the distance from the point (P) to the ellipse's center (O), i.e., the OP distance is given by

$$OP = [x^2 + y^2]^{½} = (a.\ b) / [(b^2 \cos^2 θ) + (a^2 \sin^2 θ)]^{½} \quad \text{(F. 17)}$$

A very important condition is that the normal at each point of the ellipse (i.e., the line perpendicular to the tangent of the ellipse at that point) is the bisector of the angle between the lines between it (P, Figure 10) and the foci of the ellipse (G and Q). That line (PM, Figure 10) defines angular relationships (*j* and *g*, in the GPM triangle), by which the angle of incidence (*i*) and the angle of refraction (*r*) at a point on the surface will then be considered.

Consequently, relationships can be established for the GPM and MPQ triangles:

$$GP / (\sin j) = PM / [\sin 180 - (j + g)] = MG / (\sin g) \quad (F.\ 18)$$

$$QP/[\sin (180 - j)] = PM/[\sin (j - g)] = MQ/ (\sin g) \quad (F.\ 19)$$

A number of developments may originate henceforth. Note that

$$GP + PQ = 2\,a \quad (F.\ 20)$$

$$GM + MQ = 2\,f \quad (F.\ 21)$$

Let us consider the values of GM and MQ from F. 18 and F. 19, respectively:

$$GM + MQ = 2\,f =$$
$$[GP (\sin g) / (\sin j)] + [PQ (\sin g) / (\sin j)] \rightarrow$$
$$2\,f = [2\,a (\sin g) / (\sin j)] \rightarrow$$
$$\boldsymbol{f / a = e = (\sin g) / (\sin j)} \quad (F.\ 22)$$

From F. 18, F. 19, and F. 20:

$$GP + PQ = 2\,a =$$
$$[(PM.\ \sin j) / \sin (j + g)] + [(PM.\ \sin j) / \sin (j - g)] \rightarrow$$
$$2\,a = (PM.\ \sin j).\ \{[1 / \sin (j + g)] + [1 / \sin (j - g)]\} \rightarrow$$
$$2\,a = (PM.\ \sin j).\ \{[\sin (j - g)] + [\sin (j + g)]\} /$$
$$\{[\sin (j - g)].\ [\sin (j + g)]\} \rightarrow (2\,a) / (PM.\ \sin j) =$$
$$[(\sin j)(\cos g) - (\cos j)(\sin g) + (\sin j)(\cos g) + (\cos j)(\sin g)] /$$
$$[(\sin j)(\cos g) - (\cos j)(\sin g)] . [(\sin j)(\cos g) + (\cos j)(\sin g)] \rightarrow$$
$$(2\,a) / (PM.\ \sin j) =$$
$$2 (\sin j)(\cos g) / \{[(\sin j)(\cos g)]^2 - [(\cos j)(\sin g)]^2\} \rightarrow$$
$$a \{[(\sin^2 j)(1 - \sin^2 g)] - [(1 - \sin^2 j).\ (\sin^2 g)]\} =$$
$$PM (\sin^2 j) (1 - \sin^2 g)^{½} \rightarrow$$
$$a \{[(\sin^2 j) -(\sin^2 j)(\sin^2 g)] - [(\sin^2 g - (\sin^2 j)(\sin^2 g)]\} =$$
$$PM (\sin^2 j) (1 - \sin^2 g)^{½} \rightarrow$$
$$a [(\sin^2 j) - (\sin^2 g)] = PM (\sin^2 j) (1 - \sin^2 g)^{½} \rightarrow$$

Replacing the sin g value of F. 22,

$$a [(\sin^2 j) - (e^2 \sin^2 j)] = PM (\sin^2 j) (1 - e^2 \sin^2 j)^{½} \rightarrow$$
$$\boldsymbol{a (1 - e^2) = PM. (1 - e^2 \sin^2 j)^{½}} \quad (F.\ 23)$$

The $(1 - e^2) = k$ set can be useful to simplify equations, as shown next. In any case, F. 23 gives the value of j according to that of PM (or vice versa). Given $\sin j = y / PM$, then

$$a (1 - e^2) = y (1 - e^2 \sin^2 j)^{½} / \sin j \rightarrow$$
$$a^2 (1 - e^2)^2 \sin^2 j = y^2 (1 -e^2 \sin^2 j) \rightarrow$$
$$(\sin^2 j) [y^2.e^2 + a^2 (1 - e^2)^2] = y^2 \rightarrow$$
$$\boldsymbol{\sin j = y / [y^2.e^2 + a^2 (1 - e^2)^2]^{½}} \quad (F.\ 24)$$

and therefore,

$$\boldsymbol{PM = [y^2.e^2 + a^2 (1 - e^2)^2]^{½}} \quad (F.\ 25)$$

Equation F. 17 (and similarly, F. 15 and F. 16) can be rewritten as a function of $k = 1 - e^2$:

$$OP = a.b / [(b^2 \cos^2 \theta) + (a^2 \sin^2 \theta)]^{½} =$$
$$a. (a^2 - f^2)^{½} / \{[(a^2 - f^2) \cos^2 \theta) + (a^2 \sin^2 \theta)]\}^{½} =$$
$$a. (a^2 - a^2e^2)^{½} / [a^2 - (a.e)^2 \cos^2 \theta]^{½} =$$
$$a. a (1 - e^2)^{½} / [a^2 (1 - e^2 \cos^2 \theta)]^{½} =$$
$$a (1 - e^2)^{½} / [(1 - e^2) + e^2 \sin^2 \theta]^{½} \rightarrow$$
$$\boldsymbol{OP = a [k / (k + e^2 \sin^2 \theta)]^{½}} \quad (F.\ 26)$$

The equation for the radius of curvature at one of the points P on the surface (R_P) is

$$\boldsymbol{R_P = (a.\ b)^2 / \{[(a \cos j)^2 + (b \sin j)^2]^{3/2}} \quad (F.\ 27)$$

which leads to the conclusion that for one of the vertices of the major axis ($j = 0°$),

$$R_A = (a^2.\ b^2) / a^3 \rightarrow \boldsymbol{R_A = b^2 / a} \quad (F.\ 28)$$

whereas for one of the vertices of the minor axis ($j = 90°$),

$$R_B = (a^2.\ b^2) / b^3 \rightarrow \boldsymbol{R_B = a^2 / b} \quad (F.\ 29)$$

The location of the curvature's center in relation to each of the points on the surface will be considered later; nevertheless, the relationships on the refraction of a curve without spherical aberration can be exemplified.

Refraction on a surface without spherical aberration

Consider an object point located at an infinite distance from a surface that separates two media, those of the incidence and of the refraction, with refractive indices n_i and n_r, respectively ($n_r > n_i$). Consider "A" a point on this surface where the incident radiation occurs and let q be a *finite* distance, from A, where the image (Q) of an object at an infinite distance is formed.This means that the considered surface is *curved* (since for a flat surface of an infinite radius of curvature , the Q image of an object at infinite distance would also form at infinite distance). Regardless of this surface' shape (spherical, elliptical, or any other), the calculation of the image's position (Q) is obtained from the premise that a radius of curvature can be attributed to the surface's point "A," equivalent to that of a spherical surface, as suggested by figure 6.

Let be R_A the radius of curvature at this point (A) of the surface. By the formula of refraction on a spherical surface, the relationship between the image's position, relative to A (i.e., q) and the radius of curvature (R_A) can be established according to the respective refractive indices (n_i an n_r) as

$$(R_A / q) = (n_r - n_i) / n_r \quad \text{(F. 30)}$$

which can be rewritten according to the focal power assigned to surface (F):

$$F = (n_r - n_i) / R_A = n_r / q$$

Hence,

$$\boldsymbol{R_A = (n_r - n_i) / F = (n_r - n_i)\, q / n_r} \quad \text{(F. 31)}$$

The *spherical non-aberration* condition of this surface presupposes that any point P of that surface has the image of the object(located at the infinite distance) formed exactly over Q, i.e., coinciding with it. The direction of the incidence of radiation on P is therefore parallel to the direction of incidence on point A. By the normal line to point P and (still) regardless of the surface's shape, the geometric relationships of this coincidence can be established (Figure 11).

Hypothesizing that G and Q are the foci of an ellipse to which P belongs, the following relationships will apply to triangles GPM and MPQ, respectively:

$$(GM / \sin r) = (GP / \sin i) \text{ and}$$
$$[PQ / \sin (180 - i)] = (MQ / \sin r)$$

Hence,

$$GP + PQ =$$
$$[(GM \sin i) / \sin r] + [(MQ \sin i) / \sin r] =$$
$$(GM + MQ) (\sin i) / \sin r$$

Since in ellipses, GP + PQ = 2 a, i.e., the sum of the distances from the considered point (P) to each focus (G and Q) equals the length of the ellipse's major axis; and GM+ MQ = 2 f. Thus,

$$2\,a = 2\,f\,(\sin i) / \sin r \rightarrow a / f = n_r / n_i$$

Since (f/ a) = e measures the ellipse's eccentricity, the fundamental relationship is established between its "shape" (described by its eccentricity, **e**) and the relationship between the refractive indices of the media considered, such that the incidence of parallel rays on *any* points on this surface (i.e., *all*) produce refracted rays that converge to a single point (Q), one of the ellipse's foci:

$$\boldsymbol{e = f / a = n_i / n_r} \quad \text{(F.32)}$$

In other words, for a curve to not exhibit *spherical aberration*, it must be elliptical and of an eccentricity that corresponds to the ratio of the refractive indices of the incidence and refractive media separated by it.

Once this condition is satisfied, *infinite surfaces* may exist, each with the specific focal length (AQ = q) that is a function of the radius of curvature of the apical point of the major axis (A), i.e., $\boldsymbol{R_A}$, whose value has already been defined (F. 30 and F. 31). Thus, in figure 11, as AG = a – f, the generically formulated focal length is the following:

$$q = AQ = AG + GQ = (a - f) + 2\,f = \boldsymbol{a + f = q} \quad \text{(F. 33)}$$

Finally, the variables of the ellipse can be determined from *q* (image focal length) or **F** (surface's focal power), conventionally taken with respect to the curvature radius of the apical point of the ellipse's major axis (R_A). Hence, for the length of the ellipse's semimajor axis (***a***),

$$a + (a.\, n_i)/ n_r = q \rightarrow \boldsymbol{a = q.\, n_r / (n_r + n_i)} \quad \text{(F. 34)}$$

$$\text{or } a + (a.\, n_i)/ n_r = n_r / F \rightarrow \boldsymbol{a = (n_r)^2 / (n_r + n_i).\, F} \quad \text{(F. 35)}$$

For the semifocal axis length (**f**),

$$f = q - a = q - [q.\, n_r / (n_r + n_i) \rightarrow$$
$$\boldsymbol{f = q.\, n_i / (n_r + n_i)} \quad \text{(F. 36)}$$

$$\text{or } \boldsymbol{f = n_r.\, n_i / (n_r + n_i).\, F} \quad \text{(F 37)}$$

For the semiminor axis ***(b)***,

$$b^2 = a^2 - f^2 =$$
$$[q.\, n_r / (n_r + n_i)]^2 - [q.\, n_i / (n_r + n_i)]^2 =$$
$$q^2\, [(n_r)^2 - (n_i)^2] / (n_r + n_i)^2 =$$
$$q^2\, (n_r + n_i)\, (n_r - n_i) / (n_r + n_i)^2 \rightarrow$$
$$\boldsymbol{b = q\, [(n_r - n_i) / (n_r + n_i)]^{1/2}} \quad \text{(F. 38)}$$

$$\text{or } \boldsymbol{b = [(n_r - n_i) / (n_r + n_i)]^{1/2}.(n_r / F)} \quad \text{(F. 39)}$$

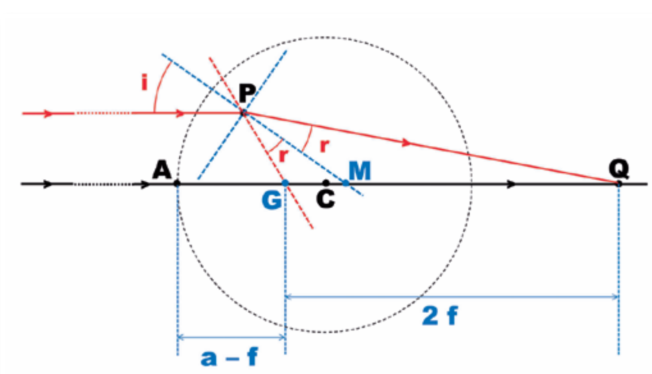

Figure 11. Refraction of parallel rays (object at infinity) from two points at the same plane of a curved surface (A and P). C is the center of curvature of an imaginary sphere tangential to A. Q is the image of the object formed on the ACQ axis. If Q is one of the foci of an ellipse, then the other is G, such that the PM line (normal to point P) is the angle bisector of GPQ.

Table 3. Formulations for the variables of a curve without spherical aberration (elliptical), depending on the values of the radius of curvature of the apical point of its major axis (R_A), or on the respective image focal power at that point (F), or on the corresponding image focal distance *(q)*.

Variable	As function of q	As function of F(*)	As function of R_A
Radius of Curvature (R_A)	$q\ (n_r - n_i) / n_r$	$(n_r - n_i) / F$	
Semimajor axis (a)	$q\ .\ n_r / (n_r + n_i)$	$(n_r)^2 / (n_r + n_i)\ .\ F$	$R_A\ (n_r)^2 / [(n_r)^2 - (n_i)^2]$
Semiminor axis (b)	$q\ [(n_r - n_i) / (n_r + n_i)]^{½}$	$(n_r / F)\ [(n_r - n_i) / (n_r + n_i)]^{½}$	$R_A\ .\ n_r / [(n_r)^2 - (ni)^2]^{½}$
Semifocal distance (f)	$q\ .\ n_i / (n_r + ni)$	$n_r\ .\ n_i / (n_r + n_i)\ .\ F$	$R_A\ .\ n_r\ .\ n_i / [(n_r)^2 - (n_i)^2]$
Radius of curvature (R_B)	$q\ (n_r)^2/(n_r+n_i)\ [(n_r)^2-(n_i)^2]^{½}$	$n^3 /(n_r+n_i)\ [(n_r)^2- (n_i)^2]^{½}\ F$	$R_A\ (n_r)^3 / [(n_r)^2- (n_i)^2]^{3/2}$
a-f	$q\ (n_r- n_i) / (n_r + n_i)$	$n_r\ (n_r - n_i) / (n_r + n_i)\ F$	$R_A\ .\ n_r / (n_r + n_i)$

(*): In considering F values (in diopters), distances are necessarily expressed in meters. In other cases, values are expressed in the same measurement unit (of *q*, or R_A).

Other relationships between these variables can also be obtained (for example, ***a - f*** or ***a / b***, etc.). Table 3 summarizes those formulations for surfaces without spherical aberration.

Some relationships are purely dependent on the ellipse itself, such as the eccentricity (e) that defines it, or others that can be deduced from the formulas already presented:

$$R_A = b^2/ a \rightarrow \boldsymbol{R_A / b = b / a} \text{ (F. 40)}$$

$$R_B = a^2/ b \rightarrow \boldsymbol{R_B / a = a / b} \text{ (F. 41)}$$

$$a / b = (R_B / a) / (b / R_A) \rightarrow \boldsymbol{R_A . R_B = a. b} \text{ (F. 42)}$$

As a numerical example, consider a surface separating two media, with refractive indices of 1.0 for incidence ($n_i = 1.0$) and 1.5 for refraction ($n_r = 1.5$). To determine the eccentricity of the (elliptic) curve such that it *does not produce spherical aberration*, $e = n_i/n_r = 1.0/1.5 = 0.666...$ Be the radius of curvature at the apical point of its major axis equals to 20 cm ($R_A = 20$ cm). The focal power of this surface (object located at an infinite distance from the surface) is F = (1.5 − 1.0) /0.2 m = 2.5 D. The image focal length of this surface is q = 1.5/ F = 1.5/2.5 = 0.6 m = 60 cm. In this curve, the semimajor axis (a) is then (calculation that can be done, alternatively, either from q = 60 cm, or from R_A = 20 cm, or from F = 2.5 D), a = 36 cm (or 0.36 m, if the calculation is done by the formula containing the F-value). The semifical axis is f = 24 cm. The semiminor axis is $b = 720^{½} \approx 26.833$ cm and the radius of curvature of the apical point of the minor axis is $R_B \approx 48.299$ cm. By Snell's law, assuming an incident radius parallel to the optical axis and tangentiating this apical point of the minor axis (a purely mathematical conjecture), we have

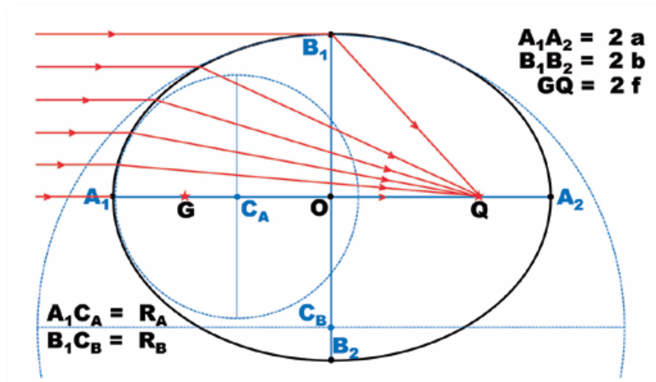

Figure 12. Graphic representation of an elliptical curve without any of the spherical aberration's characteristic elements. A_1 and A_2, poles of the major axis; B_1 and B_2, poles of the minor axis; G and Q: ellipse's foci; O: center of the ellipse; C_A: center of the osculating circle of point A_1; $A_1C_A = R_A$, radius of curvature of point A_1; C_B: center of the osculating circle of pole B_1; $B_1C_B = R_B$, radius of curvature of point B_1 . Note that at this point (B_1) the refracted ray corresponds to an incidence of 90°, and therefore, tan r = tan 41.8103° = (OQ = f) / (OB_1 = b); or sin 41.8103° = f/a; or, also, cos r = b/a. (The scale of representation maintains the variables' actual proportions.)

$$(\sin i) / (\sin r) = (1 / 1.5) = f / a$$

for f = 24 cm and a = 36 cm, expressing the eccentricity ratio of this curve. Obviously (a − f) = 12 cm, whose calculation can be confirmed by any formulas presented for the variable (a - f) in table 3.

Table 4 summarizes values of the variables of curves without spherical aberration, according to the refractive index n_r, in case the medium of incidence is air ($n_i = 1.0$).The calculations for the above mentioned example (q = 60 cm and Figure 12) correspond to those in the third row of table 4 ($n_r = 1.500$).

A very important relationship is the eccentricity of the curve, such that it does not show spherical aberration when the incidence medium is air. In this case, $n_i = 1.0$ and the expression of eccentricity is

Table 4. Elements of an elliptical curve without spherical aberration, depending on the surface's image focal length *(q)* and on the index of refraction of the refractive medium (n_r), when n_i = 1.0.

n_r	e	R_A	R_B	f	a	b
1.333	0.750	0.250 q	0.864 q	0.429 q	0.571 q	0.378 q
1.400	0.714	0.286 q	0.834 q	0.417 q	0.583 q	0.408 q
1.500	0.667	0.333 q	0.805 q	0.400 q	0.600 q	0.447 q
1.600	0.625	0.375 q	0.788 q	0.385 q	0.615 q	0.480 q
1.700	0.588	0.412 q	0.779 q	0.370 q	0.630 q	0.509 q
1.800	0.556	0.444 q	0.773 q	0.357 q	0.643 q	0.535 q
1.900	0.526	0.474 q	0.771 q	0.345 q	0.655 q	0.557 q
2.000	0.500	0.500 q	0.770 q	0.333 q	0.667 q	0.577 q

expressed by the inverse of the index of refraction of the refractive medium (n_r), that is, e = $1/n_r$. Reciprocally, $e^{-1} = n_r = a / f$. This is an interesting simplification for it shows the perfect linearity between the index of refraction n_r and the relationship between the lengths of the major axis (2 a) and the interfocal distance (2 f) of the surface without spherical aberration (Figure 13).

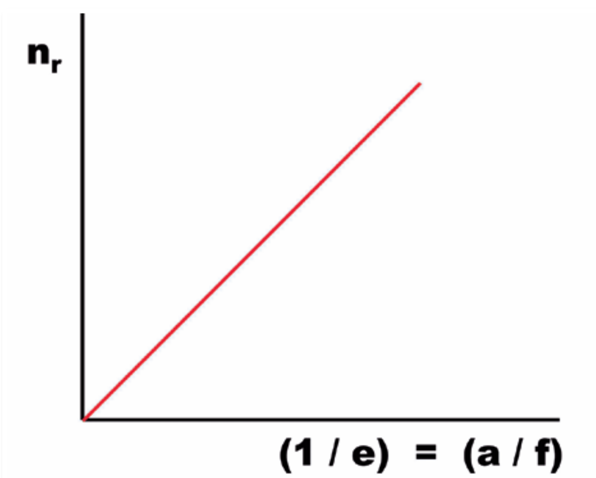

Figure 13. The eccentricity of an ellipse (e) which does not show spherical aberration is reciprocally related to the index of refraction of the refractive medium (n_r) when the incidence medium is air (n_i = 1.0).

Positioning the center of curvature

Figure 14 shows the procedures that graphically determine the position of the center of curvature of any point on an elliptical surface:

1) Draw the tangent line (TT) to the point of the surface (P) which center of curvature is to be determined.
2) Draw the *normal line NN* perpendicular to the surface tangent (TT) at the point considered (P). NN will contain the center of curvature of the surface at that point (P).
3) At the crossing point of the normal line (NN) with the major axis of the ellipse (A_1A_2), a point (M) is defined. By M, is drawn a line parallel to the tangent (TT), i.e., the SS line.
4) From the crossover point of the SS line with the line that joins point (P) to the farthest focus (Q), line PQ, i.e., point U, is drawn a perpendicular to the PQ direction (considered to be the refracted ray), line ZZ.
5) At the intersection of line ZZ with the normal line to point P (line NN), the center of the osculating circle to point P (point C_p) is located. $PC_p = R_p$ is the radius of curvature of that circle.

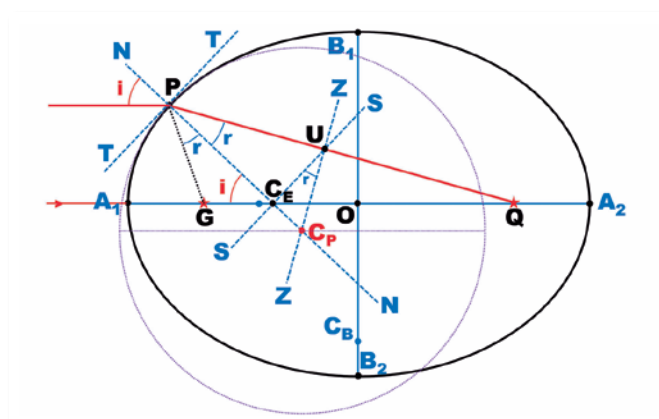

Figure 14. Pictorial representation of the process for determining C_P, the center of the osculating circle (in violet, dotted line) to point P. From U, the meeting place of a line (SS), parallel to the tangent (TT) to the point P, with the direction PQ (assumed, *incidentally,* to be that of the refracted ray), a perpendicular is drawn (ZZ), which when meeting the normal (NN) to the considered point, determines the center C_P. Thus, the radius of curvature corresponding to point P has the length PC_P.

Figure 15 is the reproduction of figure 14, purged from the elements used to determine point C_P. It makes it easy to understand the following geometric deduction.

The coordinates of the point P and C_P are P (x, y) and C_P (u, h) , respectively. Hence, the radius of curvature (R_P) is determined as

$$PC_P = \boldsymbol{R_P = [(y + h)^2 + (x - u)^2]^{1/2}} \quad \text{(F. 43)}$$

The relationships between the functions of the elliptical curve (a, b, f, R_A), the coordinates of one of its points (x, y, θ) and the respective functions related to refraction (i, r, n_i, n_r, q, F, etc.) the following formulations are obtained:

$$\boldsymbol{PM = y / \sin i} \quad \text{(F.44)}$$

From the PGM triangle, by the law of sines,

$$(GM / \sin r) = PM / \sin (i + r) = y / (\sin i) \sin (i + r)$$

From the MPQ triangle, by the law of sines,

$$[PQ / \sin (180 - i)] = (MQ / \sin r)$$

Yet,

$$\sin (i - r) = y/PQ$$

Hence,

$$MQ = y. (\sin r) / (\sin i) \sin (i - r)$$

Finally, since GM + MQ = 2 f,

$$2 f = y.(\sin r) / \{ [1 / (\sin i) \sin (i + r)] + [1 / (\sin i) \sin (i - r)] \} \rightarrow$$
$$2 f / y (\sin r) =$$
$$\{[(\sin i) \sin (i-r)] + [(\sin i) \sin (i+r)]\} /\{(\sin^2 i)[(\sin (i+r)].[\sin (i-r)]\} \rightarrow$$
$$2 f / y (\sin r) =$$
$$\{[\sin (i-r)] + \sin (i+r)]\} / \{(\sin i)[(\sin (i+r)].[\sin (i-r)]\} \rightarrow$$
$$[2 f (\sin i)] / y (\sin r) =$$
$$[2 (\sin i) (\cos r)] / [(\sin^2 i).(\cos^2 r) - (\cos^2 i).(\sin^2 r)] \rightarrow$$
$$f/y (\sin r) =$$
$$(\cos r) / [(\sin^2 i). (\cos^2 r) - (\cos^2 i). (\sin^2 r)]$$
$$\boldsymbol{f / y = (\sin r). (\cos r) / [(\sin^2 i) - (\sin^2 r)]} \quad \text{(F. 45)}$$

Given (f/ a) = (sin r)/(sin i),

$$\boldsymbol{\rightarrow a / y = (\sin i). (\cos r) / [(\sin^2 i) - (\sin^2 r)]} \quad \text{(F. 46)}$$

Alternatively, the equation can be converted to the calculation of ***i*** (or ***r***) according to **y** and ***a*** (or ***f***, or θ, etc.). Thus, from F. 45,

$$f^2 / y^2 = [(n_i \sin i)/n_r]^2 \{1 - [(n_i. \sin i)/n_r]^2\} / \{[(\sin^2 i) - [(n_i. \sin i)/n_r]^2\} \rightarrow$$
$$\boldsymbol{\sin i = (y. n_i. n_r) / \{f^2 [(n_r)^2 - (n_i)^2]^2 + (n_r)^4. y^2\}^{1/2}} \quad \text{(F. 47)}$$

E.g., for n_r = 1.5, n_i = 1.0, f = 24 cm, and for y for θ = 40° (by F. , being a = 36 cm and b = 720½ cm,

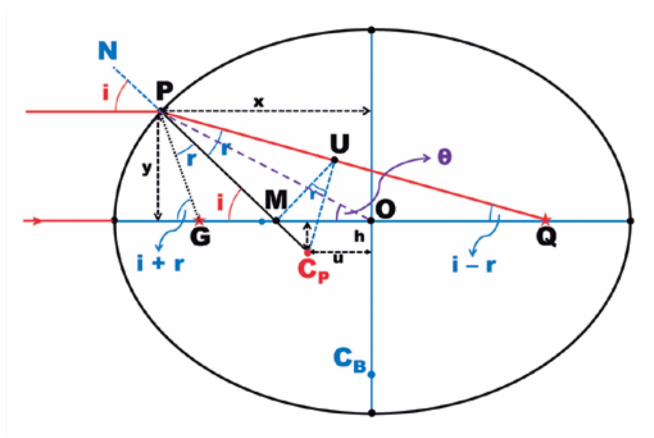

Figure 15. Illustration of an elliptical curve whose center of curvature of a point P (x, y) is C_P (u, h).

it comes y = 20.06115396 cm), results sin i = 1.5 y / $(900 + y^2)^{1/2}$, and thus, i = 56.49204167°. (Note that this result can also be obtained from F. 24, when j = i.)

The value of the "apparent" radius of curvature (PM = R_M, Figure 15 and F. 44) can then be calculated as PM = R_M = 20.06115396 / sin 56.49204167° = 24.05964716 cm (result also identical to that obtained by F. 25). The true radius of curvature ($PC_P = R_P$) is easily calculated from the PUC_P and PMU triangles, respectively:

$$UP = R_P. \cos r = PM / \cos r \rightarrow \boldsymbol{R_P = R_M / \cos^2 r} \quad \text{(F. 48)}$$

from which R_P = 34.81831666 cm.

Hence, the values of the coordinates of the center of curvature (***u*** and ***h***) can be derived as follows:

$$\cos i = (x - u) / R_P \rightarrow \boldsymbol{u = x - R_P. \cos i} \quad \text{(F. 49)}$$

$$\sin i = [y - (- h)] / R_P \rightarrow \boldsymbol{h = (R_P. \sin i) - y} \quad \text{(F. 50)}$$

Table 5 shows approximate values (up to the third decimal place) of the values of the coordinates of a point P (x, y) depending on the angle (θ), those of the line (OP) between that point (P) and the center of the ellipse (O), the coordinates of the center of curvature (u and h), the "apparent" radius of curvature value (PM = R_M) and the true (R_P) value as well as the angle of incidence (i) and that of refraction (r) corresponding to the points which coordinates are considered.

Table 6 complements table 5, showing (in the penultimate column) values of the coordinate (m,0) of point M, i.e. the "pseudo" center of curvature of point P, that is, the place where the normal at that point

Table 5. Calculated values for the angle of incidence (i) and of refraction (r) on a surface without spherical aberration, separating the media of incidence (n_i = 1.0) and of refractionj (n_r = 1.5), for the points which Cartesian (x and y) and/or polar (θ) coordinates are considered. Also shown are the Cartesian coordinates of the respective centers of curvature of each of these points (u and h), as well as the corresponding radius of curvature.

θ	x	y	$(x^2+y^2)^{½}$	$(x - x_C)$	$(y - y_C)$	x_C	y_C	R_M	R_P	i	r
0°	36,000	0	36,000	20,000	0	16,000	0	20,000	20,000	0	0
10°	35,033	6,177	35,573	20,288	6,439	14,745	−0,262	20,420	21,285	17,609	11,635
20°	32,349	11,774	34,425	20,740	13,588	11,609	−1,814	21,485	24,795	33,231	21,429
30°	28,460	16,432	32,863	20,555	21,361	7,906	−4,930	22,804	29,645	46,102	28,711
40°	23,908	20,061	31,210	19,222	29,032	4,686	−8,971	24,060	34,818	56,492	33,771
50°	19,089	22,750	29,698	16,704	35,832	2,386	−13,082	25,100	39,534	65,007	37,174
60°	14,230	24,648	28,460	13,242	41,285	0,988	−16,637	25,884	43,356	72,216	39,406
70°	9,426	25,897	27,559	9,138	45,194	0,287	−19,297	26,421	46,109	78,569	40,802
80°	4,691	26,604	27,014	4,656	47,526	0,035	−20,922	26,731	47,753	84,405	41,567
90°	0	26,833	26,833	0	48,299	0	−21,466	26,833	48,299	90,000	41,810

Table 6. Values of the sum and of the difference of angles of incidence *(i)* and of refraction *(r)*, distances to each of the poles (d_1 and d_2), respective sums ($d_1 + d_2$) and ratios (d_2 / d_1), distances from images formed on the major axis measured from the anterior apical point *(s)*, "apparent" radii of curvature (R_m), and the coordinates of the corresponding centers of curvature *(m)* for incidences of rays originating from objects at infinity on points of a surface without spherical aberration, which separates the media of incidence (n_i = 1.0) and of refraction (n_r = 1.5), wherein Cartesian (*x* and *y*) and/or polar (θ) coordinates are considered.

θ	x	y	(i + r)	(i – r)	d_1	d_2	$d_1 + d_2$	d_2 / d_1	m	R_m	s
0°	36.000	0	0	0	12.000	60.000	72.000	5.000	16.000	20.000	60.000
10°	35.033	6.177	29.244	5.974	12.645	59.355	72.000	4.694	15.570	19.463	60.000
20°	32.349	11.774	54.659	11.802	14.434	57.566	72.000	3.988	14.377	17.972	60.000
30°	28.460	16.432	74.813	17.392	17.026	54.974	72.000	3.229	12.649	15.811	60.000
40°	23.908	20.061	90.263	22.721	20.061	51.939	72.000	2.589	10.626	13.282	60.000
50°	19.089	22.750	102.181	27.833	23.274	48.726	72.000	2.094	8.484	10.605	60.000
60°	14.230	24.648	111.622	32.810	26.513	45.487	72.000	1.716	6.325	7.906	60.000
70°	9.426	25.897	119.370	37.767	29.716	42.284	72.000	1.423	4.189	5.236	60.000
80°	4.691	26.604	125.972	42.839	32.873	39.127	72.000	1.190	2.085	2.606	60.000
90°	0	26.833	131.810	48.190	36.000	36.000	72.000	1.000	0	0	60.000

crosses the major axis of the curve (its main *optical axis*), given by the equation

$$m = x - PM \cos i \quad \text{(F. 51)}$$

or of its respective distance to the anterior apical point (that is, the distance $A_1M = R_m$, Figure 14). Note that R_m corresponds to the value of the *true* radius of curvature, when i = 0° ($R_m = R_A$) and the value R_m = PM cos i, when 90° > i > 0°. In fact, for these cases, m + R_m = x.

It also shows some variables supporting other possible calculations, as well as their relationships. In fact, in certain cases, it may be interesting to know the value of the distance from each focus to the point P (where incidence and refraction occurs), that is, GP = d_1 and QP = d_2 (Figures 14 and 15), as well as the result of its sum (always equal to the ellipse's major axis, i.e., $d_1 + d_2$ = 2 a = 72.000 cm, in the example considered) and of its ratio (d_2/d_1), incidentally, equal to the ratio between the sine of the sum and of the difference between the angles of incidence (i) and of refraction (r) They may be deduced from figure 15:

$$y = [PG\ (= d_1)] \sin (i + r) = [PQ\ (= d_2)] \sin (i - r)$$
$$\rightarrow d_2 / d_1 = [\sin (i + r)] / [\sin (i - r)] \quad \text{(F. 52)}$$

Finally, it considers the *s* value, which as expected, must correspond to distance A_1Q (Figure 14), that of the surface's image focal length :

$$s = (a - x) + [y / \tan (i - r)] \quad \text{(F. 53)}$$

E.g.: for θ = 40°, the corresponding values are (Table 6): x = 23.908 cm, y = 20.061 cm, (i - r) = 22.721°, from which we obtain s = 60.000 cm.

The eccentricity of the curve of no spherical aberration as a contingency of the object position

The fact that the eccentricity of an elliptical curve justifies the absence of spherical aberration in the refraction of the image formed by it from incident rays parallel to the optical axis does not mean that this property applies to other incidences, i.e., to other positions of the object in relation to that surface. In fact, if an object located at an infinite distance forms an image 60 cm from the surface, as exemplified, this corresponds to an anterior apical curvature radius of the surface (R_A) equal to 20 cm, which conditions all geometric variables of the curve (a = 36 cm, f = 24 cm, b = $720^{½}$ cm, whose relationships are typified by the eccentricity e = 0.666…, as demonstrated for media with refractive indices n_i = 1.0 and n_r = 1.5). Meanwhile, an object at any other distance from this apical point will necessarily form its image subject to the laws of refraction, at different distances, as per the expressions F. 06 or F. 07.

$$\boldsymbol{s = q = (n_r.\ p.\ R_A) / [(n_r - n_i).\ p - n_i.\ R_A]} \text{ (F. 54)}$$

Thus, under the pre-established conditions for the surface without spherical aberration of apical curvature radius R_A = 20 cm, n_r = 1.5, and n_i = 1.0, we get p = 120 cm, q = 90 cm. On the same elliptical surface previously used as an example, the incidence of another incident rays of this object to other points of it (such as P, Figure 16) leads to algebraic relationships, which allow the calculation of the distance ***s*** at which the image is formed.

For the LBP triangle, being LA = p, AB = (AO - BO) = (a - x), PB = y (taken from the θ = 40° value), angle w is determined:

$$\boldsymbol{\tan w = y / [p + (a - x)]} \text{ (F. 55)}$$

from which one obtains w = 8.636 cm. To determine the *direction* of the refracted ray (angle r), the value of the angle of incidence *(i)* is required. The latter can be determined from the *inclination* of the apparent radius of curvature (PM = R_M) or of the true (R_P) relative to the ellipse's major axis, i.e., i – w. The apparent radius of curvature of point P is known (PM = R_M). Hence,

$$PM = \boldsymbol{R_M = y / \sin (i - w)} \text{ (F. 56)}$$

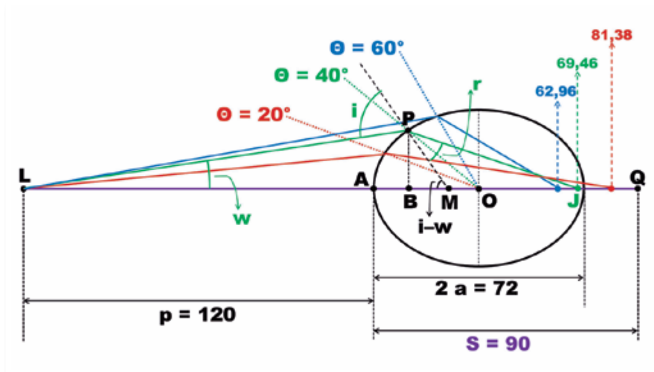

Figure 16. Elliptical surface with *no spherical aberration* for the incidence of rays parallel to its major axis (optical axis of the surface), but which shows it for incidences from a finite-distance light source (L). Note that, as in the case of a circular (or spherical) surface, the farther the distance of the incident ray from the optical axis, the closer to the (elliptical) refractive surface the image forms.

therefore (i – w) = 56.492°, hence i = 65.128° which, by rhe Snell's law gives r = 37.217°. BQ is calculated next, as follows:

$$\boldsymbol{\tan (i - r - w) = y / BJ} \text{ (F. 57)}$$

This results tan 19.275° = 20.061/BJ; thus, BJ = 57.365 cm. Finally, for the desired unknown value, s = AJ = AB + BJ:

$$\boldsymbol{s = (a - x) + [y / \tan (i - r - w)]} \text{ (F. 58)}$$

that is, s = 12.092 + 57.365 = 69.457 cm. For other y-values (corresponding to other supposedly known values of the coordinate θ), the calculations are analogous, leading to the values shown in figure 16.

In short, the curve whose shape (eccentricity) is suitable for nullifying the spherical aberration of images of an object located at infinite distance is unable to do so if the object is at a finite distance. For a closer object, a curve with a *greater eccentricity* would be required. In other words, "asphericity" (condition of no spherical aberration) (which can be defined by the eccentricity value of the curve) is not applicable for any distance. Moreover, spherical aberration is not caused by a presumed *spherical* surface, but rather it is a property inherent to the very nature of refraction.

Therefore, despite occurring on spherical surfaces, the term "spherical aberration" is absolutely inappropriate since it also occurs on elliptical surfaces, even those which can be defined as "aspherical" for objects at infinite distances, but do not keep such a property for images of closer objects. A more convenient label for such a type of refractive deffect might be of *surface (or diopter) aberration* or of aberration of *surface (diopter) curvature*. Curvature aberration can simply lead to confusion with "field curvature," which are aberrations of other phenomena.

Calculation of the eccentricities that maintain asphericity at finite distances

The relationship between the cancelation of "surface" (diopter) aberration ("spherical" aberration), caused by (1) a certain eccentricity of the curve (elliptical, as has been studied)—in turn dependent on the refractive indices of the medium separated by the surface—and (2) the object's distance, which images must form on a single point (image "focus"), regardless of the incidence on the surface, can be sought. In fact, it is possible to assume that "asphericity" could be also obtained for images of objects located at finite distances with curves of greater eccentricities. For an object at a given distance (for example, p = 120 cm), located on the optical axis of *any* curve that separates medium with refractive indices $n_i = 1.0$ and $n_r = 1.5$ and has a radius of curvature $R_A = 20$ cm at its apical point (coincident with the optical axis), i.e., with a focal power $(n_r - n_i)/R_A = 2.5$ D, and considering a null incidence (i = 0°) it will form the respective image, also on the optical axis, always, at the *same* distance (s = 9). In any case, although infinites of these curves are possible, only one elliptic will also have the sum $a + f = 9$, thus extracting a specific value for its eccentricity, which can then be calculated.

If L is the considered light source and if Q is the point upon which the image of the refraction produced by the null incidence is formed, Q will be one of the foci of the elliptical curve. Thus, the distance from this source (L) to the anterior "pole" (vertex) of the ellipse (A), LA = *p*, is the *distance from the object* to the surface; and AQ = $a + f$ is the distance at which the respective image is formed. (Points L and Q are supposed to be coaxial to the ellipse's major axis, or optical axis of the surface.) By the law of refraction,the paraxial ray equation:

$$\boldsymbol{F = (n_r - n_i) / R_A = (n_i / p) + [n_r / (a + f)]} \quad \text{(F. 59)}$$

Variables R_A (radius of curvature of the anterior apical point of the ellipse), *a* (its major semiaxis) and *f* (its semifocal distance) are thus automatically related to the variables related to refraction, n_i (index of refraction of the incidence medium), n_r (index of refraction of the refractive medium), *p* (distance from the object to surface), or *F* (focal power of the surface). The distance from the image to the anterior pole of the surface *(s)* is automatically determined (s = a + f). In fact, this equation (F. 59) can be rearranged:

$$[F - (n_i / p)]\,(a + f) = n_r \rightarrow (a + f)\,(F.\,p - n_i) = n_r.\,p \rightarrow$$
$$[p.\,(n_r - n_i) / R_A] - n_i = (n_r.\,p) / (a + f) \rightarrow$$
$$[p.\,(n_r - n_i)\,a / b^2] - n_i = (n_r.\,p) / (a + f)$$

But given , $b^2 = a^2 - f^2$

$$[p.\,(n_r - n_i).\,a] - n_i\,(a^2 - f^2) = n_r\,.\,p\,(a - f) \rightarrow$$
$$-\,p.\,n_i\,.\,a - n_i\,(a^2 - f^2) = -\,n_r\,.\,p.\,f \rightarrow$$
$$\boldsymbol{p\,(n_r.\,f - n_i.\,a) = n_i\,(a^2 - f^2)} \quad \text{(F. 60)}$$

More concisely,

$$p.\,[n_r.\,(a.\,e) - n_i.\,a] = n_i\,(a^2 - a^2e^2) \rightarrow$$
$$\boldsymbol{p.\,(n_r.\,e - n_i) = n_i.\,a\,(1 - e^2)} \quad \text{(F. 61)}$$

Further, depending on R_A, since

$$b^2 = a^2 - f^2 = a^2\,(1 - e^2)$$
$$p.\,(n_r.\,e - n_i) = n_i.\,a\,(b^2 / a^2) \rightarrow$$
$$p.\,(n_r.\,e - n_i) = n_i.\,b^2 / a$$
$$\rightarrow \boldsymbol{p.\,(n_r.\,e - n_i) = n_i.\,R_A} \quad \text{(F. 62)}$$

In fact, for objects at infinity (p = ∞), $n_r.\,e = n_i$, as already demonstrated. Thus, for the object at infinity (p = ∞) the eccentricity of the "aspherical" curve (e) equals the reciprocal of the refractive indices of refraction on the incidence (n_i) and refractive (n_r) media. For distance p = 120 cm, R_A = 20 cm, n_i = 1.0, and n_r = 1.5, the eccentricity of the curve is

$$120\,(1.5\,e - 1.0) = 1.0 \times 20 \rightarrow$$
$$9.0\,e - 6.0 = 1.0 \rightarrow e = 7/9 = 0.777\ldots$$

which is greater than the eccentricity of the spherical curve for the object at infinity (e = 1/1.5 = 2/3 = 0.666...). This eccentricity (e = 2/3) is therefore not appropriate to nullify the spherical aberration for images of an object situated at other distance.

The value of this ellipse semimajor axis (a) is determined by:

$$a\,(1 - e^2) = R_A$$

i.e., $a = 20/[1 - (0.777)^2\,] = 50.625$ cm; hence f = 39.375 cm and b ≈ 31.820 cm.

However, for other incidences on this surface, considering the distance of 120 cm (= p) from the object to the front vertex of the major axis of such an elliptical surface, spherical aberration *still persists*, as table 7 shows (second column). Finally, to possibly nullify the "spherical" aberration (or curvature aberration) corresponding to an object at a finite distance, this curve with that "typical" eccentricity ($e = 7/9$) shows nothing special compared to others (e.g., with $e = 2/3$, as seen in Figure 16). It only represents the *maximum* eccentricity value, such that $a + f = 90$ cm and $R_A = 20$ cm. In fact, for an ellipse of greater eccentricity, such as $e = 0.8$, with $a = 50$ cm and $f = 40$ cm, then $R_A = b^2/a = (a^2 - f^2)/a = 18$ cm.

For this ellipses family (with $a + f = 90$ cm and $R_A = 20$ cm) *other eccentricities* may be specific to values of *other incidences*, as table 7 shows. Remember that for a finite distance and a *non-axial* incidence, the applicable formula for the calculation where the distance at which the image *(s)* is formed equals $a + f = a + ae = a\ (1 - e)$ is F.5:

$$\boldsymbol{n_r / n_i = \{[R^2 + (s\text{-}R)^2 + 2\ (s\text{-}R).(R\text{-}\ y^2)^{½}]^{½}. (p+R)\} / \{[R^2 + (p+R)^2 - 2\ (p+R).(R^2\text{-}y^2)^{½}\ ]^{½}\ (s\text{-}R)\}} \quad \text{(F.63)}$$

in which the value of the radius of curvature *(R)* varies according to the point of incidence but with a specific value (corresponding to *y*), such that the image forms at distance $s = 90$ cm.

The results in table 7 are paradigmatic:

a) The second column corresponds to a *circular* curve (zero eccentricity, flatness 0). Note that the more peripheral the incidence point (the greater the angle θ value), the closer to the surface the corresponding image forms (red numbers), which means the so-called *positive* "spherical" aberration.

b) Curves of greater eccentricity (and greater flatness) will progressively correct this aberration. E.g.: for eccentricity $e = 37.8858/ 52.1142$ (seventh column from the left) the aberration is corrected to the incidence of 40° (the image forms at exactly 90.000 cm, as desired). For this inclination (40°), greater eccentricities will produce *positive* aberrations (red colored numbers) and smaller eccentricities will produce *negative* aberrations (blue colored numbers). (Exception made to the circle, second column, where $e = 0/20.0 = 0$.)

c) Elliptical curves can't prevent the "spherical aberration" of images of objects located at a finite distance. At best, a curve with a certain eccentricity promotes the coincidence of the position of the image produced by null incidence (in this case, 90 cm from the surface) only for another incidence. The table shows the eccentricities of the curves corresponding to incidences of 10° to 70°, although curves of intermediate flatness between those shown are also possible. E.g.: between the 0.3651 flatness curve (which matches the 10° incidence) and the 0.3499 (which matches the 20° incidence), an intermediate flatness curve will promote the formed image at 90 cm from the surface for a certain specific incidence, between 10° and 20°.

d) The curve of a given eccentricity produces *negative* aberrations for incidences lower than that which promotes the desired position of the image (at 90 cm); and *positive* aberrations for incidences greater than the specific.

e) Thus, a curve promoting total asphericity (images positions always at 90 cm) should have a *decreasing* eccentricity (and/or flatness); in this case, starting from $e = 7/9$, third column, with an increasing angle of incidence. *It would not be an elliptical curve, therefore.*

f) Even if such a curve could then be constructed, it would serve to neutralize the aberration of "sphericity" for, only and specifically, the position of the object at $p = 120$ cm (and mediums of incidence and refraction such as those aforementioned). For other distances and other indices of refraction , a new curve specific to each case should be constructed.

In fact, the fundamental equation by which one can understand the relationship between the geometric properties of an elliptical curve and the basic law of refraction (by Snell and Descartes) is that of F. 22, that is, $e = (\sin g) / (\sin j)$, where $j = i - w$ (for finite-distance objects) and $g = r$, the angle of refraction obtained from this incidence. In other words, the equivalence between the angle of incidence *(i)* and that of refraction *(r)* - which must always obey Snell's law- is also subordinated to the ellipse's geometry with its specific *(e)* eccentricity. Despite not explicit in table 7, this F. 22 is only satisfied at a specific angle $(i - w)$, corresponding to an incidence *(i)* which relation to the angle of refraction *(r)* expresses the eccentricity of the curve (e) value. Tables 8 and 9 show values of some of the variables of the ellipse geometry for different angles of incidence *(i)* relative to the ellipse's center (θ), when *w* values also vary, for ellipses with eccentricities $e = (f / a) = 38.8584 / 51.1416 = 0.7598$ (Table 8) and $e = 37.8858 / 52.1142 = 0.7270$ (Table 9). Note the highlighted (red) values of $s = 90.0000$ (the position of the image satisfied, in each case, by refraction for angles $\theta = 20°$ in Table 8; and angles $\theta = 40°$ in Table 9) and those of the respective relations of the sines of the angles (i − w) and r, coinciding with the value of the corresponding eccentricity of the curve (e = 0.7598 in Table 8; 0.7270 in Table 9), respectively. In other cases, this coincidence is inexistent and the image position does not occur at the desired point (s = 90.0000 cm).

Figure 17 also shows the relationship between an ellipse's variable $k = 1 - e^2$ and the polar (angular) coordinate of a point (measured from its center), which refracted ray therein (from an incident ray originating from a finite-distance object) reaches the optical axis in the same place where this object's image corres-

Table 7. Position of Image of an object situated at p = 120 cm from elliptical surfaces with dioptric power = + 2.5 D (R_A = 20 cm, index of refraction of incidence medium n_i = 1.0 and of refractive medium, nr = 1.5) of different eccentricities (e = f/a), with incidences to points of different polar coordinates (θ) measured from the center of the ellipse, where ***a*** is the value of the semimajor axis, ***f*** the focal semidistance and 1 − k½ its flatness.

1-k½ f a θ	0 0 20.000	0.3715 39.3750 50.6250	0.3651 39.2275 50.7725	0.3499 38.8584 51.1416	0.3315 38.3873 51.6127	0.3133 37.8858 52.1142	0.2963 37.3823 52.6177	0.2804 36.8845 53.1155	0.2657 36.3916 53.6084
0°	90.000	90.000	90.000	90.000	90.000	90.000	90.000	90.000	90.000
10°	88.270	87.085	90.000	97.526	107.626	119.031	131.217	144.053	157.605
20°	83.444	81.618	85.378	90.000	98.082	107.158	116.785	126.836	129.922
30°	76.450	77.596	79.365	83.931	90.000	96.767	103.878	111.143	118.814
40°	68.419	75.686	77.029	80.464	84.997	90.000	95.203	100.519	105.948
50°	60.357	75.199	76.242	78.840	82.351	86.123	90.000	93.915	97.867
60°	52.972	75.497	76.332	78.437	81.152	84.075	87.041	90.000	92.954
70°	46.658	–	–	–	–	–	–	–	–
80°	41.543	–	–	–	–	–	–	–	–

Table 8. Values of geometric variables at the incidence point of an elliptical curve with focal power + 2.5 D and 0.7598, which separates media with refractive indices n_i = 1.0 and n_r = 1.5, for an object at a distance p = 120 cm.

θ	y	x	w	i – w	i	r	s	sen r / sen (i–w)
10°	8.7032	49.3584	4.0877	22.6444	26.7321	17.4504	97.5256	0.7789
20°	16.2420	44.6244	7.3155	40.7322	48.0477	29.7226	90.0000	0.7598
30°	22.0777	38.2397	9.4318	53.7924	63.2243	36.5258	83.9309	0.7376
40°	26.2830	31.3228	10.6462	63.2645	73.9108	39.8331	80.4639	0.7172
50°	29.1882	24.4918	11.0699	70.4722	81.5421	41.2554	78.8399	0.6997
60°	31.1283	17.9719	11.4876	76.2861	87.7737	41.7716	78.4371	0.6857

Table 9. Values of geometric variables at the incidence point of an elliptical curve with focal power + 2.5 D and e = 0.7270, which separates media with refractive indices n_i = 1.0 and n_r = 1.5, for an object at a distance p = 120 cm

θ	y	x	w	i – w	i	r	s	sen r / sen (i–w)
10°	8.9004	50.4765	4.1849	20.5042	24.6891	16.1685	119.0310	0.7950
20°	16.7592	46.0456	7.5723	37.6658	45.2382	28.2526	107.1582	0.7747
30°	23.0295	39.8881	9.8800	50.7626	60.1425	35.5246	96.7667	0.7502
40°	27.6939	33.0043	11.2592	60.6677	71.9269	39.3291	90.0000	0.7270
50°	31.0063	26.0173	11.9821	68.4144	80.3965	41.0961	86.1234	0.7069
60°	33.2660	19.2061	12.2737	74.7718	87.0455	41.7422	84.0753	0.6900

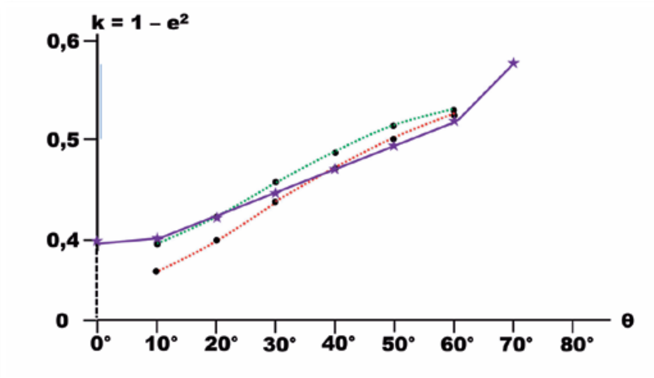

Figure 17. Graphical representation of the relationships between the ellipse's variable k = 1 – e^2 and the angle of incidence relative to the ellipse's center (θ), must be such that, by refraction, the image forms 90 cm away from the surface as per table 7 data (purple line). Note the almost perfect linearity between 10° and 60° positions for k values correcting spherical aberration, at each considered angle but in a line *inclined* in relation to the axes instead of as ideally desired: a straight line parallel to the axis of the abscissas, as it would be by the conception of a "total" asphericity. The green and red dotted lines correspond to table 8 (angle of 20°) and table 9 (angle of 40°) results, respectively. They show "deviations," i.e. "aberrations of sphericity," in relation to the points where this aberration should not exist.

ponding to zero incidence is formed. Note that this ellipse variable in which this situation occurs is greater the more peripheral the point where the refraction is considered (purple line), i.e., for each angle taken from the center of the ellipse, an *increasing* variation of k (as a function of the angulation value) would be adequate to satisfy such conception. Contrarily, for this desired asphericity to occur for any point on the curve, the representation of a single *k* value would be that of a straight horizontal line (perpendicular to the ordinate axis and parallel to the abscissas axis).

The different positions of the curvature centers of each point of the surface and their refractometric consequences

One of the peculiarities of the elliptical curve is that there is a specific center of curvature for each of its points. Its derivation was outlined in figure 14. The calculation of its coordinates, as well as the value of the actual curvature radius (R_P = PC_P) and "apparent" (R_M = PM) curvature radius were explained by figure 15 and by F.49 and F. 50 (for coordinate values), F. 27 and F. 43 (curvature radius R_P values by polar and Cartesian coordinates, respectively), F. 44 (apparent radius of curvature values PM = R_M) and F. 48 (values of the relationship between R_P and R_M).

The geometric location of the various centers of curvature, from the anterior apical point (A), R_A (F. 28) to the vertex of the minor semiaxis (B), R_B (F. 29) and their relationships (F. 40-42), crossing the intermediate curvature centers, is the so called *ellipse's evolute* (Figure 18).

In ellipses, the points of greatest curvature (A_1 and A_2, radii of curvature A_1A' and A_2A," Figure 18), have the centers of the respective osculating circles (i.e., the centers of curvature of these points) *within* the ellipse, whereas the points of smallest curvature (B_1 and B_2, curvature radii B_1B' and B_2B," Figure 18) have the centers of the respective osculating circles (i.e., the centers of curvature of these points) *either outside* or inside the ellipse (see Figure 12). The ellipse's evolute is analytically described by equation:

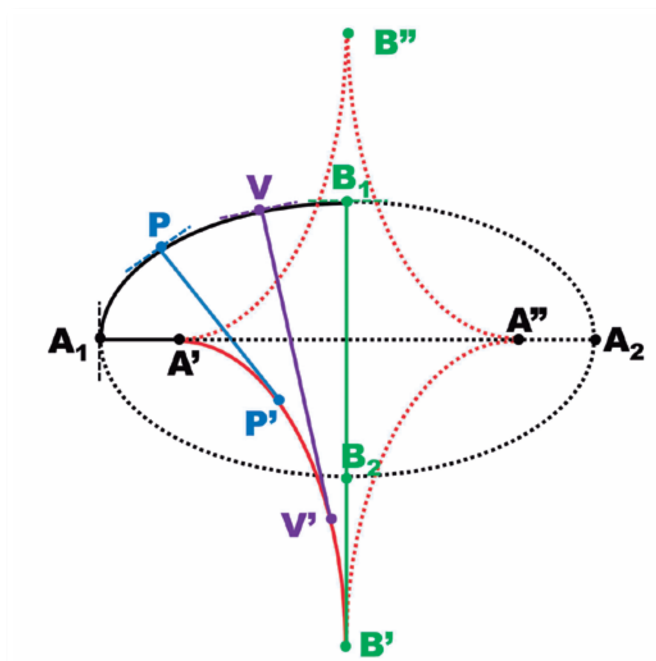

Figure 18. Graphic representation of an ellipse and the locus of the centers of the oscillating circles of each of its points (*evolute*), i.e., of the locus of the centers of curvature of each one of them. Considering the second quadrant (above and to the left), in which the ellipse is represented by a solid black line, the point A_1 (vertex of the semimajor axis) has its center of curvature at A' (the radius of curvature is AA '); point B_1 (vertex of the minor semiaxis) has its center of curvature at B' (radius of curvature is B_1B'); and between them, points P and V, have their respective centers of curvature at P' (the radius of curvature is PP') and V' (radius of curvature is VV'). The red solid line between A' and B' is the corresponding evolute of this quadrant. Note that each radius of curvature is normal (perpendicular) to the considered point of the ellipse and tangent to the evolute. (The figure's proportions are strictly maintained on the scale, where A_1O = a = 7.2; B_1O = b = 4.0; therefore, A_1A' = R_A = b^2/ a = 2.22 and B_1B' = R_B = a^2/b = 12.96.)

$$\boldsymbol{[(a.\ x)^{2/3} / (a^2 - b^2)^{2/3}] + [(b.\ y)^{2/3} / (a^2 - b^2)^{2/3}] = 1} \quad \text{(F. 64)}$$

The radius of curvature (R_P) value of any point of an ellipse can be determined by both F. 27 and F. 48. It always requires the knowledge of two of the ellipse's variables (*a, b, f, or e*, through which the other two are calculated) as well as defining the point for which the respective radius of curvature is required. This raises the need to know two of their coordinates (the Cartesian, *x* and *y*, or one of them plus the angular coordinate θ). For example, by replacing the value of

$$b^2 = a^2 - f^2 = a^2 - (ae)^2 = a^2 (1 - e^2) \text{ in F.27:}$$

$$R_P = a^2.\ a^2 (1 - e^2) / [a^2 \cos^2 j + a^2 (1 - e^2) \sin^2 j]^{3/2} \rightarrow$$
$$R_P = a^4 (1 - e^2) / [a^2 - a^2 e^2) \sin^2 j]^{3/2} \rightarrow$$
$$R_P = a^4 (1 - e^2) / a^3 [1- e^2 \sin^2 j]^{3/2} \rightarrow$$
$$\boldsymbol{R_P = a (1 - e^2) / (1- e^2 \sin^2 j)^{3/2}} \quad \text{(F. 65)}$$

In fact, if j = 0°, $R_P = R_A = a (1 - e^2) = a.\ (b^2/a^2) = b^2 / a$, ratifying the F.28. If j = 90°, from F. 65 it comes $R_P = R_B = a (1 - e^2) / (1 - e^2)^{3/2} = a / (1 - e^2)^{½} = a / (b^2/a^2)^{½} = a / (b/a) = a^2/b$, ratifying the F. 29.

From F. 48 and F. 44, the angles *i* and *r* may be replaced by their respective generic versions, j and g, where (sin g)/(sin j) = e, hence: $y = R_M.\ \sin i = (R_P \cos^2 r).\ \sin i = R_P (1 - \sin^2 r).\ (\sin i) = R_P (1 - \sin^2 g).(\sin j) \rightarrow$

$$\boldsymbol{R_P = [\ y / (1 - e^2 \sin^2 j).\ \sin j]} \quad \text{(F. 66)}$$

Or, still, by F. 25 ($PM = R_M$) and F. 48:
$$R_P = R_M / \cos^2 r = R_M / (1 - e^2 \sin^2 j) \rightarrow$$

$$\boldsymbol{R_P = [a^2(1 - e^2)^2 + e^2y^2]^{½} / (1 - e^2 \sin^2 j)} \quad \text{(F. 67)}$$

Thus, for example, for the ellipse with a = 50.625, e = 7/9 and θ = 10°, from F. 16 it comes y = 8.594755891 . Hence, by F. 25, R_M = 21.08759556, and thus, j = arc sin (y / R_M) = 24.05252876°. Finally, either by F. 65, F. 66, or F. 67, we can determine R_P to be 23.44343249.

Table 10 shows the radius of curvature (R_P) values for points of this ellipse, with different angular coordinates relative to its center (θ), as well as those of other corresponding variables. Figure 19 depicts the variation of these curvature radii and the corresponding dioptric values (for a curve separating media with indices of refraction n_i = 1.0 and n_r = 1.5).

It is easy to understand from figure 19 that the dioptric values of this aspherical elliptic curve progressively reduces as the incidence (parallel to the optical axis) becomes more peripheral (blue line).

Table 10. Values of radii of curvature (R_P) for points with different angular coordinates (θ) of an ellipse with eccentricity e = 7/9, semimajor axis, a = 50.625. The values of auxiliary variables are also shown: *y* (shortest distance from the considered point to the main axis of the ellipse), R_M (value of the "apparent" radius of curvature), i.e., distance between the considered point of the surface and the optical axis (on the bisector of the angle between its lines to the foci) and *j* (angle of the radius of curvature with the optical axis, equal to the angle of incidence, when the incidence is parallel to the optical axis).

θ	y	R_M	j	R_P
0°	0	20.000	0°	20.000
10°	8.595	21.088	24.053°	23.443
20°	15.945	23.533	42.654°	32.582
30°	21.526	26.082	55.617°	44.360
40°	25.467	28.149	64.788°	55.759
50°	28.145	29.651	71.660°	65.174
60°	29.911	30.679	77.151°	72.191
70°	31.018	31.337	81.817°	76.937
80°	31.626	31.703	86.015°	79.659
90°	31.820	31.820	90.000°	80.544

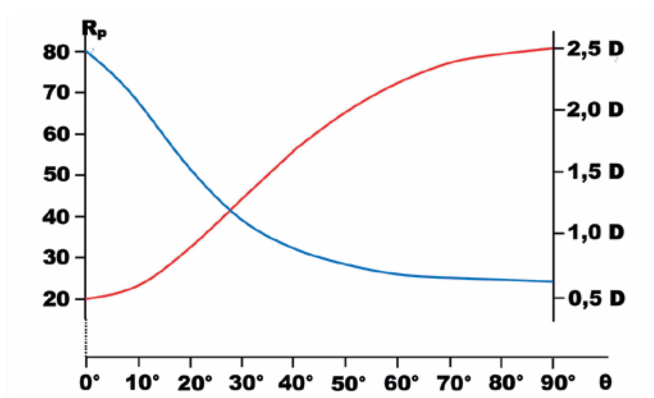

Figure 19. Representation of the radius of curvature (R_P) in centimeters (blue line) and the respective dioptric values (red line), for each polar coordinate (angle θ) for an ellipse with eccentricity e = 7/9, when the refractive indices of refraction for the media of incidence and of refraction are 1.0 and 1.5.

It should be noted, however, that a *prolate* curve is being considered, i.e., with the major axis coincident with the horizontal plane. If the elliptical curve is rotated 90° around an axis perpendicular to the plane of its representation, passing through its center, its radii of curvature and respective centers do not change relative to the ellipse. Instead, it assumes an *oblate* shape, i.e., with the minor axis coinciding with the horizontal (Figure 20) and therefore with *decreasing* radii of curvature and respective *increasing* dioptric values as more peripheral incidences occur.

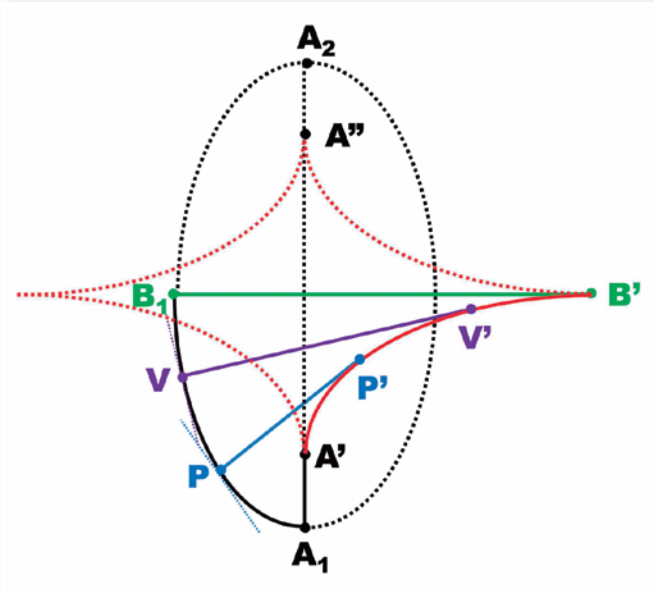

Figure 20. Ellipse identical to that on Figure 18, differing from it in that it was rotated 90° counterclockwise, around an axis perpendicular to the plane considered, crossing the center of the figure. Note that all proportions and relationships are maintained.

The multifocality of elliptical curves

As the dioptric values of a lens at its different points are directly proportional to those of its respective curvatures, i.e., the greater the curvature of a surface at a given point (smaller the radius of curvature therein), the greater its respective dioptric power, the elliptical curves naturally lend themselves to satisfy multifocality, according to the incidence and/or the direction of the refracted ray; in other words, according to the direction of the visual axis that crosses it.

In fact, by choosing a dioptric value (i.e., a curvature) for a given point on a surface, it is possible to set a new curvature (greater or lesser) for another point, such that it features a transition of the corresponding dioptric values (progressive or regressive) between them. More common is to want a certain dioptric value for a "far" distance (e.g., +1 D) and a higher one for a "near" distance (e.g. + 4 D), which usually occurs at downward gaze positions. In a myopic person, the case would be a variation from -5 D to -2 D, for example, but *regressive* variations (or *semiprogressive*, different names to basically express the same concept) may also be convenient. Although it is not this dissertation's aim to discuss the convenience and applications of such clinical concepts, nor the technical difficulties of operating their fabrications, the principles that govern this multifocality (much desired in ophthalmological practice) will only be addressed briefly.

Consider an elliptic curve which represents the first (or anterior) principal plane of a lens, i.e., the imaginary place where all the refraction of incident (and refracted) rays takes place, whose effect would be of a decreasing *multifocal* dioptric power in a *prolate* curve (Figure 18), or crescent in an *oblate* (Figure 20). Due to the greater applicability of increasing multifocalities, the example will be restricted to this concept and shape (*oblate*) of the ellipse, whose explanation will be developed based on figure 21.

Let be R_1 the retinal (foveal) image of the object O_1 (located at infinite distance) formed by the "lens" (the abstract surface A_1B_1). The dioptric power of this surface at point B_1 is assumed to correct a possible optical error of the ocular system, so that its value can be empirically determined. Note that the visual axis (line $R_1C\,E_1\,O_1$) is coincident to the radius of curvature (B_1B') at point B_1, i.e., is tangent to the curve's evolute at B′. Another line tangent to the ellipse's evolute at P′ may be made coincident to a new position. of the visual axis to fixate object O_3 (line $R_3CE_3O_3$), so that PP′ is the radius of curvature of the surface at point P′, i.e., the point where the visual axis to fixate the object O_3 (line $R_3CE_3O_3$) cross the surface. The optical power of P may be chosen to allow the required optical correction for objects (O_3) at "near distances" (the so-called "addition" over the required optical correction for "far distances"). Therefore, if

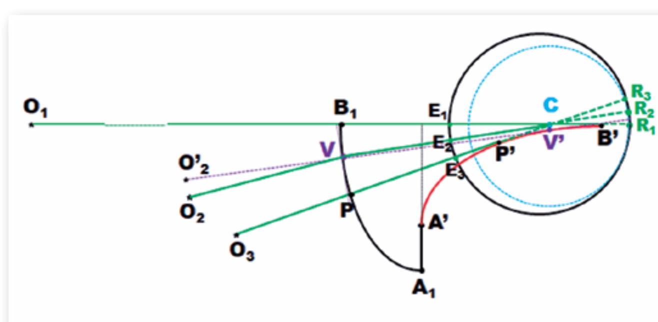

Figure 21. Schematic representation of an elliptical surface (A_1B_1) and of its respective evolute (red line, between points A' and B', the centers of rotation of points A_1 and B_1, respectively) before an eye (black circle) whose center of rotation is C (a little behind the actual geometric center of the eye). E_1R_1, E_2R_2 and E_3R_3 are the successive positions of the visual axis for fixation of objects O_1, O_2 and O_3, respectively. For simplification, the visual axis was supposed to cross the center of ocular rotation (C).

the surface's dioptric power values for points B_1 ("far distances") and P ("near distances") are chosen, that means, if radii of curvatures of surface's points B_1 and P are known (as function of variables n_r and n_i), the curve between B_1 and P is also determined.

Intermediate surface's points between B_1 (the longer radius of curvature, therefore the lesser dioptric power) and P (the shorter radius of curvature, therefore the greater dioptric power) will have intermediate radius of curvature and, therefore, intermediate dioptric powers, so that a surface's "multifocal" effect results. Note, however, that as B′ and P′ are tangents to the curve's evolute and coincident to the optical axis, the tangent point to the evolute at any other intermediate point (as V′, which corresponds to the surface's point V), lies *below* the center of rotation C (for which passes the visual axis). Therefore, the visual axis (R_2CE_2V) and the surface's radius of curvature at V (= VV′) although having a common point of coincidence (V) are not collinear. The angle between them (R_2VV') is equal to the angle of refraction (r) at V, so that there exists at this same point (V) an angle between the directions of the actual position of the object in space (O_2), line VO_2 and its perceived one (O'_2), line VO'_2, the angle of incidence (i).

Although the rationale of the multifocality of an elliptical curve is so relatively simple, the calculations are complex and will not be here presented.

Appendix: sphere, spheroids, and ellipsoids

Refractions occur on surfaces, or on entities of three-dimensional space that can be analytically described by specific values in each of its *axes.* Circles, ellipses, and other curves are "figures," or flat sections (in two-dimensional space) of realities (in three-dimensional space). In other words, they are merely pictorial representations (lines) of solids which *surfaces* (three-dimensional) separate the media in which refractions occur. Thus, it is appropriate to extend the study on the "realities" of which these figurations now addressed (elliptical curves) are mere simplified (planar) representations

In ophthalmology, it is customary for the three spatial axes to be named as "z" (vertical), "y" (horizontal, longitudinal, i.e., coincident with the visual axis) and "x," also horizontal, but perpendicular to the "y" axis, separating "left" and "right," or "lateral" (or "temporal") and "medial" (or "nasal"), mutually orthogonal, i.e., each axis belonging to two planes and perpendicular to the remainder. Thus, although the structures may have any sizes, they can be represented by their proportional dimensions on each of these axes.

A volume represented by three axes of equal dimensions (x = y = z) is a *sphere.* If two of them are equal but different from the third, the possibilities are six (x = y > z; x = y < z; x = z > y ; x = z < y; y = z > x; y = z < x) and the represented volumes are *spheroids* (Figure 22). In fact, they are solids generated by the rotation of an ellipse, around one of its axes (Figure 23) and, therefore, the sections perpendicular to that axis of revolution will *always* be represented by circles (hence the name "spheroids"). Finally, when the three axes are described by different values, the possibilities are six more (x > y > z; x > z > y; y > x > z; y > z > x; z > x > y; z > y > x) and solids called *scalene ellipsoids.* (Figure 24, showing *one* of them).

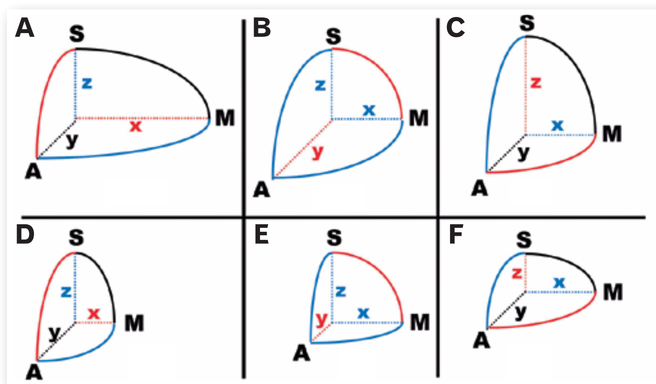

Figure 22. Illustration of spheroids: (a): prolate (x > y = z); (b): prolate (y > x = z); (c): prolate (z > x = y); (d): oblate (z = y > x); (e): oblate (x = z > y); (f): oblate (y = x > z).

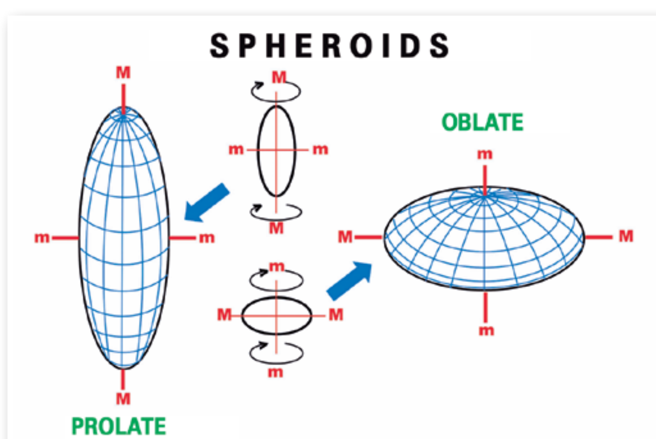

Figure 23. Representation of spheroids, three-dimensional figures resulting from the rotation of an ellipse around its major axis, MM (generating a *prolate* spheroid), or smaller, mm (generating an *oblate* spheroid).

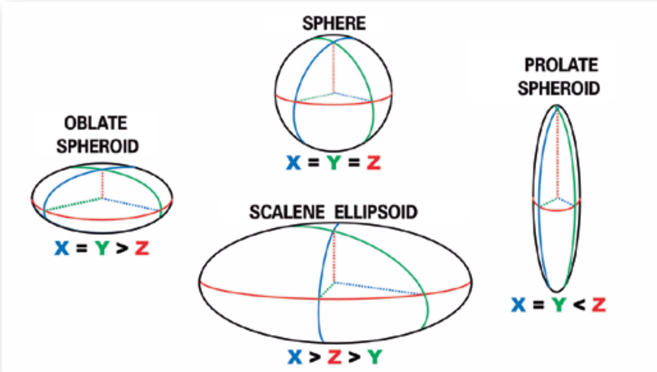

Figure 24. Representation of solids according to the length of their orthogonal axes: equal (sphere), two equal and one longer (prolate spheroid) or shorter (oblate spheroid), or three unequal (ellipsoids), of which one (x > z > y) is shown.

Note that in the *prolate* form (similar to a *kibbeh*, a rugby ball, or a football) one of the axes is *longer* than the two others (equal) and in the *oblate* form (similar to a *dragee*, or a thickened disc at its center, like a "flying saucer"), one of the axes is *shorter* than the other two (equal). However, always considering the relative position of the axes as coincident to that of the eye (y, the longitudinal, or anteroposterior; z, the vertical, or super-inferior; and x, the transverse, or latero-medial), the differences between these presentations appear. For example, in *prolates* by the disposition of the major axis (their "acute peaks") and in oblates by the disposition of the minor axis (their " flat poles").

Thus, when the longer axis is y (longitudinal), the vertical (between A and S) and horizontal (between A and M) curves will be elliptical and equal to each other (Figure 22b) and regardless of the surface appearance (like, for example, the cornea) is "conical", there is no astigmatism (difference in dioptric powers between them). Thus, also, if the *minor* axis is the y (longitudinal), the vertical curve (between A and S) and the horizontal (between A and M) will be elliptical and equal to each other (Figure 22 e); the surface appears "conical" in these two planes (or meridians) but also without astigmatism. When the major or minor axis is x, the transverse (respectively, Figures 22 a and 22 d), there is a difference between the curve of the horizontal plane (between A and M, elliptical) and the vertical (between A and S, circular), therefore causing astigmatism. In figure 22 a, the plane (or meridian) of lesser curvature (greater curvature radius, lesser dioptric power) is horizontal, configuring an astigmatism "with-the-rule"; in figure 22d, the greater curvature (smaller curvature radius, greater dioptric power) is horizontal, representing an example of "against-the-rule" astigmatism. Thus, also, when the major or minor axis is the vertical (z, respectively figures 22 c and 22 f) there is a difference between the curve in the vertical plane (between A and S, elliptical) and the horizontal (between A and M, circular), so that figure 22 c represents an "against-the-rule" astigmatism and figure 22 f represents an astigmatism "with-the-rule". Although it cannot be guaranteed (since in the composition of ametropias the axial factor is relevant), it is at least allowed to conjecture that errors related to curvatures induce refractometric errors; therefore, astigmatism: myopics in their excesses (in oblate spheroids, Figures 22, d, f), or hyperopics in their insufficiencies (in the prolate spheroids , Figures a, c). Remember that, in these astigmatisms, the pointed *prolate* form, the longer axis is vertical (Figure 22 c) or transverse (Figure 22 a); and in the oblate, or *flat* form, too (Figures 22 d and 22 f, respectively), as opposed to the condition in which the z and x axes appear equal (Figures 22 b and 22 e), which raises apparently paradoxical interpretations regarding the respective curvatures considered.

SYNOPSIS

Introduction: Knowledge about refraction is shown to be essential in ophthalmic practice. Refraction is a physical phenomenon dependent on the change in the speed of propagation of radiant energy in its transition from a medium from where it originates (the medium of incidence) to another, by which it follows (the medium of refraction), governed by a law of great simplicity, which dictates the relation of such *speeds* (or, equivalently, relative numbers that represent them, the specific refractive indices of each of these media), and the *directions* in which the radiant energy wavefronts propagate. The measurement of these directions is specific to each point of the surface of separation between the media of incidence and of refraction and depends on an imaginary line perpendicular to that point (its *normal*). Although refraction is a *punctual* phenomenon, only considering the relation of *directions* of the incident and refracted radiant energy, it is, in broader terms, dependent on the *surface* to which such a point belongs.

Curvature of a surface: The characterization of a point on a surface is determined by its *curvature*, a concept that is quantified by the length of a straight line, the *radius of curvature* of the point in question. In geometric optics, although the *direction* in which the image of an object is formed depends only on Snell's law, its *position* is subjected to the radius of curvature of the point where the refraction takes place. The radius of curvature is infinite on flat surfaces, finite and constant in spheres (or circles), finite and variable in other curves (such as conical sections). *All* corrective lenses for ocular optical defects are formed by curved surfaces (at least one). Ocular refractometry texts address refraction on flat (prisms) and spherical (lens) surfaces and are limited to them. Refraction on spherical surfaces is inherently defective (spherical aberrations). Spherical aberration-free (aspherical) and multifocal lenses require the development of optics based on *varying radii of curvature* at different points on the refracting surface.

Refraction on spherical surfaces: The principles and formulations of this type of refraction are reviewed for the generic case of the position of the image of an object point located at a finite distance. Based on these principles, the *paraxial ray* equation, that of when the *object* is located at an infinite distance, and that of the basic formulation of the surface dioptric power are derived. Despite the relative simplicity of its study, refraction on spherical surfaces has as an intrinsic consequence: *spherical aberration.* This corresponds to the distribution of incident radiant energy, for example, in a beam of parallel rays (flat wave fronts) over an extensive region of space where the various crossings of the various refracted rays occur, delimited by a surface (the *caustic*).

Aspherical Surfaces: The ones concentrating in a single point (the image focal point, when the incidence comes from infinity) all the refracted radiant energy. Their design requires that the farther from the optical axis the incidence is, the smaller the radius of curvature of the surface becomes, when compared to that of the surface's vertex (located on its optical axis). This notion corresponds to what occurs in ellipses (and other curves), which shapes can accentuate the spherical aberration of spherical surfaces (aberrations called *"positive"*), reduce it and even neutralize it (by an aspherical elliptical surface) or invert it (producing *"negative"* spherical aberrations).

The semantic question of asphericity: The different ways in which ellipses and their varied results can be presented, raises the discussion about the inappropriateness of the term "spherical" aberration, as it does not depend exclusively on spherical surfaces and may occur in elliptical surfaces (in a "positive" or "negative" way). It is proposed to use the terms of *superaberrant* surfaces for those which produce "positive" aberration, greater than that of the spherical aberrations of a sphere, *subaberrant* surfaces when they stil produce "positive" aberrations, but lesser than that of spherical surfaces), *inaberrant* when aberration is eliminated (the aspherical surfaces themselves) and *contraberrant* if they produce *negative* aberrations. With this terminology, any oblate surface will be *superaberrant*, whereas among the prolates, *sub-aberrations* ("positive"), *inaberrations*, or *contraberrations* ("negative") may be found.

The geometry of ellipses: The geometric properties of the ellipses, their terms and relationships are succinctly presented.

Refraction on a surface without spherical aberration: Based on the geometric properties of the ellipses, the condition under which the refraction of an incident ray parallel to the optical axis on one (any) of its points gives a refracted ray that meets the optical axis at one of the foci (the *focal point surface image*) is examined. It is possible to formulate that, in these cases (incidence of planar wavefronts), asphericity is achieved when the eccentricity of the ellipse (e) equals the ratio of the indices of refraction of the incidence (n_i) and refraction (n_r) media, i.e., $e = n_i/n_r$. Equations showing the relationships between the different surface properties and the values of its dioptric power are developed as well as the eccentricity values so that the surface be *inaberrant* (aspherical) according to the index of refraction of the medium of refraction, when that of incidence is air ($n_i = 1.0$).

Positioning the center of curvature: We present the method by which the position of the center of curvature of any point of an ellipse can be determined graphically, as well as the analytical method for knowing its coordinates, the value of the respective radius of curvature and other relationships between the properties of these "aspherical" ellipses.

The eccentricity of the spherical non-aberration curve as a contingency of the object's position: It is shown that the curve for which "asphericity" is obtained for the incidence of planar wave fronts (objects situated at infinity, rays parallel to the optical axis) does not maintain this quality for objects at finite distances, i.e., the eccentricity of the asphericity promoting curve is a function of the distance from the radiation source.

On the calculation of the eccentricity that maintains asphericity at finite distances: The fact that the shape (eccentricity) of the curve that provides asphericity for the formation of the (unique) image of an object located at infinite distance does not maintain this quality to the object's position at other distances, does not mean that a relationship cannot be found between a specific eccentricity and the distance from the source. This possibility is studied, and we find that there is in fact an appropriate eccentricity such that for *a specific incidence*, the position of the image of an object located at the finite distance coincides with that obtained from the definition of the focal power of the surface (i.e., with that of the null incidence). However, for smaller and larger angles of incidence on the curve that ensures this "pairing", the aberrations remain *negative and positive*, respectively. Likewise, for a certain incidence, curves with smaller eccentricities produce *negative* aberrations and those with larger eccentricities will produce *positive* aberrations. The *variability of eccentricity* necessary to produce the coincidence of images of an object located at finite distance, *depending on the angle of incidence*, means that an elliptical curve (single eccentricity) cannot satisfy that condition (asphericity for images of objects situated at finite distances). objects).

The different positions of the curvature centers of each surface point and their refractometric consequences: The study of the positions of the centers of curvature of the different points of an elliptic curve is resumed to describe their proper "geometric place" , a new curve, the ellipse's *evolute* and what occurs when this ellipse is rotated around an axis perpendicular to its figure, making it pass from one "form" (prolate) to another (oblate). Although the eccentricity (descriptive of the "formal" relationships of the ellipse) remains the same, its spatial arrangement (with the longest axis horizontally or vertically) will transform decreasing variations of curvatures (increasing their respective radii, into curves called *prolates*) into increasing ones (*oblates curves*).

The multifocality of elliptical curves: The variability of the values of the curvature radii of an elliptical surface (defined by its eccentricity) is a matrix of what refractometry calls "multifocality." Thus, increasing ("progressive") or decreasing ("regressive") dioptric power lenses are simple variations of *elliptical* shapes (eccentricities) (even though calculations for proper adjustments to the required dioptric values may be complex).

Appendix: spheres, spheroids, and ellipsoids: Ellipses are geometric curves resulting from the oblique sections of bodies (such as cones, cylinders, toruses), whose tridimensional extensions include spheroids and ellipsoids. These are described in their conceptions (ellipse's revolutions around its axes) and applications (e.g. in the definition of astigmatism).

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